Useful quantum states in the presence of noise in a Bose Josephson junction

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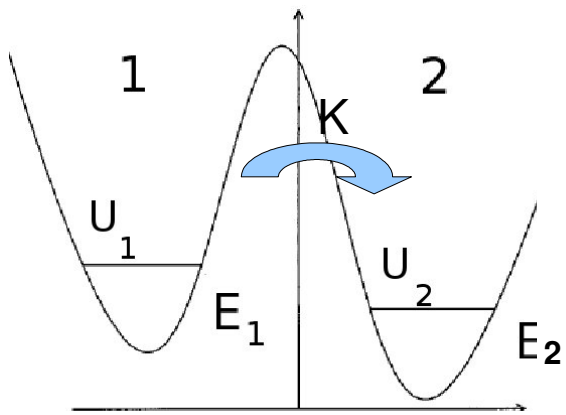
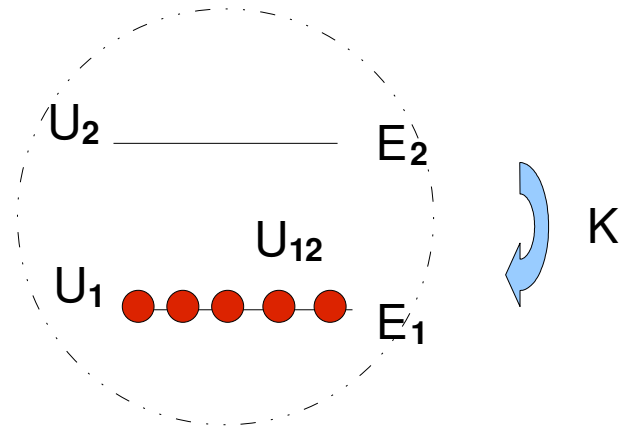
Outline

- Bosonic Josephson junctions
- Useful entangled states for atom interferometry
- Quenched unitary dynamics
  - Squeezed states;
  - Macroscopic superpositions;
- Quenched dynamics in the presence of noise
  - Effect on squeezed states?
  - Effect on macroscopic superpositions?
  - Most useful states for interferometry?

Experimental realizations of a Bose Josephson junction (BJJ)





$$\hat{H}^{(0)} = E_1 \hat{a}_1^\dagger \hat{a}_1 + E_2 \hat{a}_2^\dagger \hat{a}_2 - K(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + U_1 \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + U_2 \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2$$

$$\hat{H}^{(0)} = E_1 \hat{a}_1^\dagger \hat{a}_1 + E_2 \hat{a}_2^\dagger \hat{a}_2 - (K \hat{a}_1^\dagger \hat{a}_2 + K^* \hat{a}_2^\dagger \hat{a}_1) + U_1 \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + U_2 \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + U_{12} a_1^\dagger a_1 a_2^\dagger a_2$$

$$\boxed{\chi = U_1 + U_2}$$

$$\lambda = (E_1 - E_2) + (N-1)\frac{U_1 - U_2}{2}$$

$$\boxed{\chi = U_1 + U_2 - 2U_{12}}$$

$$\hat{J}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \equiv \hat{n}$$

$$\hat{J}_y = -i\frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{J}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

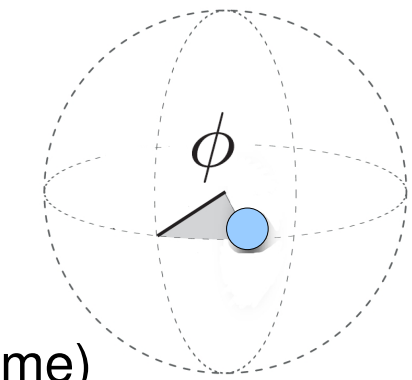
$$N = a_1^\dagger a_1 + a_2^\dagger a_2$$

$$J^2 = N/2(N/2+1)$$

$$\boxed{\hat{H}^{(0)} = \chi \hat{J}_z^2 - \lambda \hat{J}_z - 2K\hat{J}_x}$$

Ground state/possible basis of a Bose Josephson junction

$$\hat{H}^{(0)} = \chi \hat{J}_z^2 - \lambda \hat{J}_z - 2K\hat{J}_x = \chi(\hat{J}_z - \frac{\lambda}{2\chi})^2 - 2K\hat{J}_x$$



$\chi \gg KN$ (Fock regime)

$|\psi_{GS}\rangle = |n\rangle$ Fock state

$\hat{J}_z|n\rangle = n|n\rangle \quad n = \mathrm{Int}\,[\lambda/(2\chi)]$

Definite atom number in each of the two modes

$\chi N \ll K$ (Rabi regime)

$|\psi_{GS}\rangle = |\phi\rangle$ Atomic phase state (equator of the Bloch sphere)

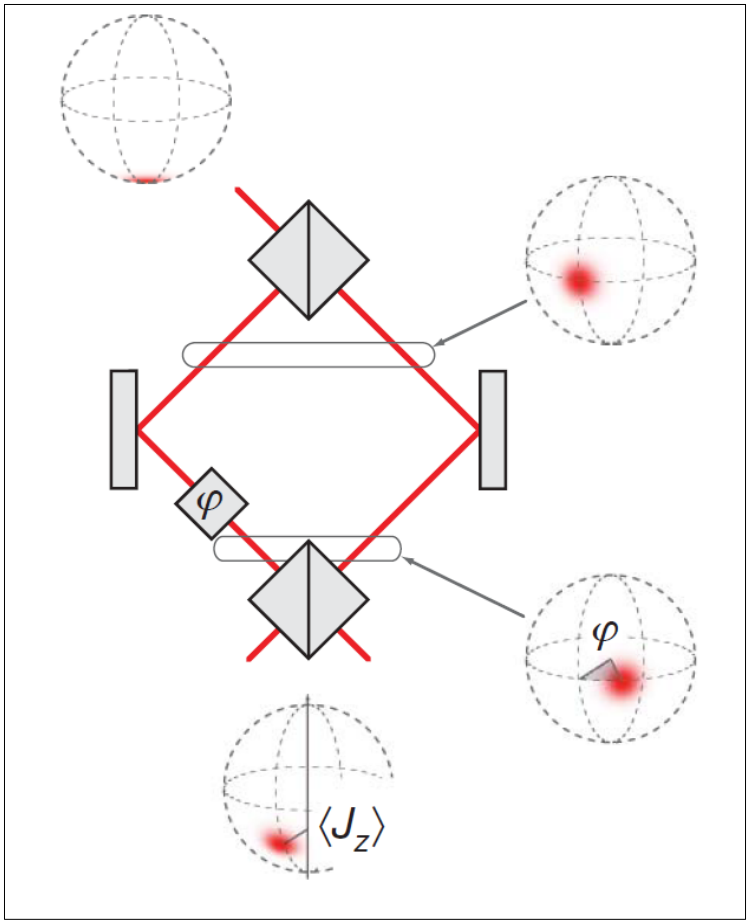
$$|\phi\rangle = \frac{1}{2^{\frac{N}{2}}} \sum_{n=-N/2}^{N/2} \binom{N}{\frac{N}{2}+n}^{\frac{1}{2}} e^{-i\phi(\frac{N}{2}+n)}|n\rangle$$

$$= \frac{(a_1^\dagger + e^{-i\phi}a_2^\dagger)^N}{\sqrt{N!}}|0\rangle$$

All the atoms are in a coherent superposition of the two modes

Ramsey atom-interferometer

Goal: to estimate a phase shift Θ with the highest possible precision.



$$|\psi\rangle_{out} = e^{-iJ_y\frac{\pi}{2}} e^{iJ_z\theta} e^{iJ_y\frac{\pi}{2}} |n=-N/2\rangle$$
$$= e^{i\theta J_x} |n=-N/2\rangle$$

$$\langle \hat{J}_z\rangle_{out} = \frac{N}{2}\cos\theta \qquad \Delta\theta = \frac{\Delta \hat{J}_{z_{out}}}{|\partial\langle \hat{J}_{z_{out}}\rangle/\partial\theta|}$$

$$\Delta \hat{J}_z = \sqrt{N}/2$$

$$\max|\partial\langle \hat{J}_{z_{out}}\rangle/\partial\theta| = N/2$$

Best possible sensitivity:

$$\Delta\theta = \frac{1}{\sqrt{N}} \equiv \Delta\theta_{SN}$$

To realize the ports: ➔ Set $\chi \sim 0$;

$$\hat{H}^{(0)} = -\lambda \hat{J}_z - 2|K|(\cos\delta \hat{J}_x + \sin\delta \hat{J}_y)$$

➔ Tune K, λ , δ and let evolve in time.

*[Wineland et al, PRA **50**, 67 (1994); Kim et al, PRA **57**, 4004 (1998); figure from C. Gross et al, Nature **464**, 1165 (2010)].*

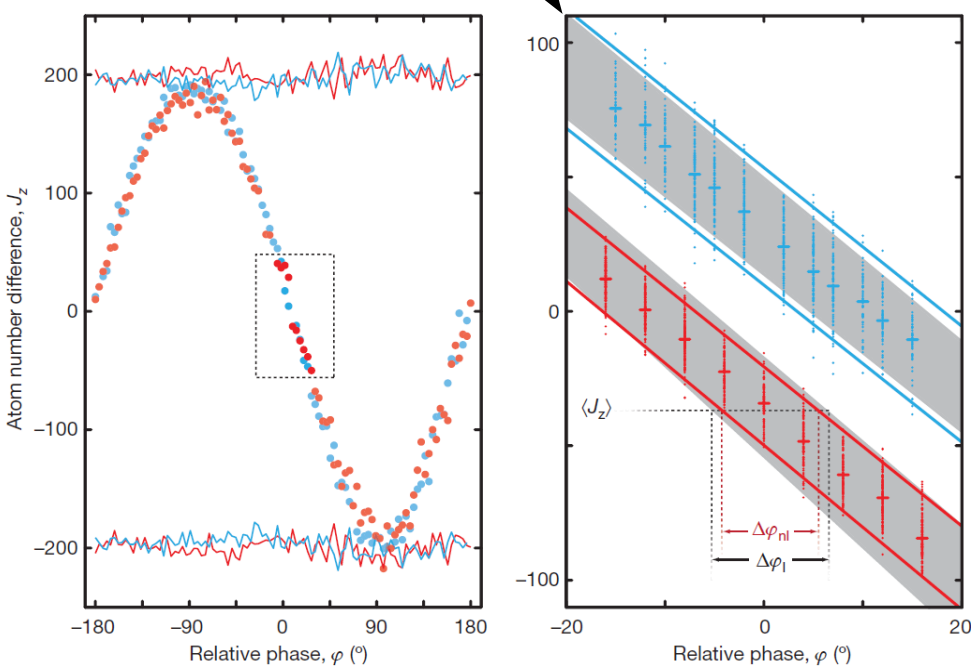
Enhanced phase resolution by squeezing

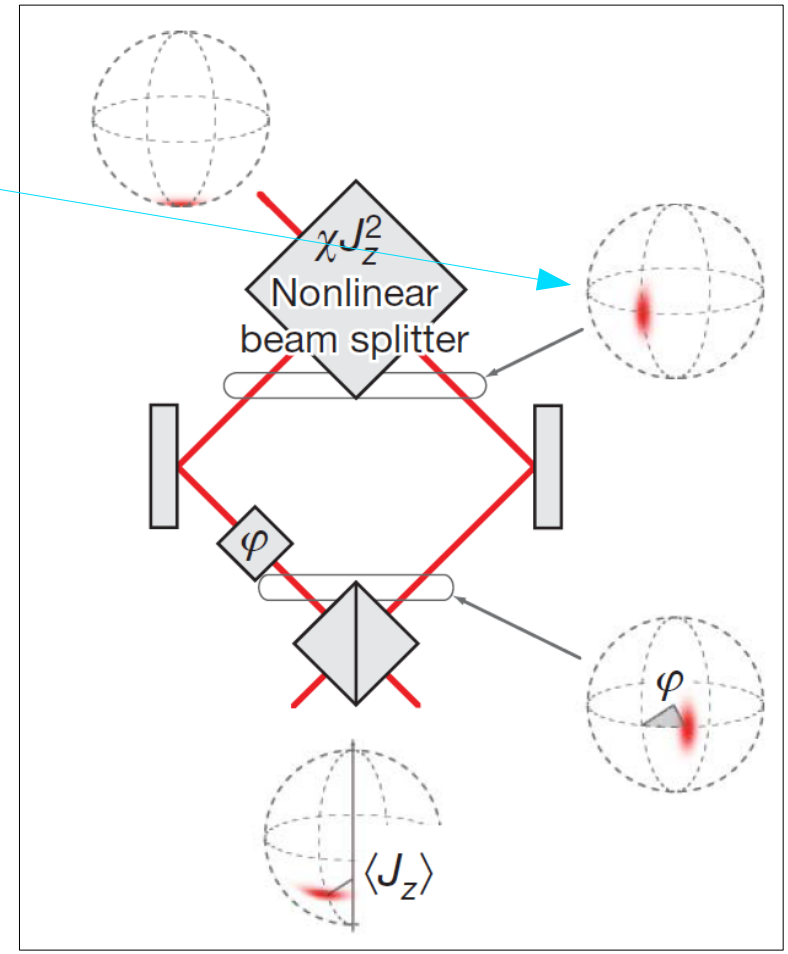
$$\Delta \hat{J}_{\vec{n}} \Delta \hat{J}_{\vec{p}_1} \geq \frac{|\langle \hat{J}_{\vec{p}_2} \rangle|}{2}$$

$$\Delta \hat{J}_{\vec{n}} < \Delta \hat{J}_{\vec{p}_1}$$

$$|\Delta\theta| = \frac{\Delta \hat{J}_{z_{out}}}{\partial \langle \hat{J}_{z_{out}} \rangle / \partial \theta}$$

$$\xi^2 = \min_{\vec{n}} \frac{N \Delta^2 \hat{J}_{\vec{n}}}{\langle \hat{J}_{\vec{p}_1} \rangle^2 + \langle \hat{J}_{\vec{p}_2} \rangle^2}$$





$$\xi^2 < 1 \quad \Rightarrow \quad \Delta\theta < \Delta\theta_{SN} \equiv \frac{1}{\sqrt{N}}$$

*[C. Gross et al, Nature **464**, 1165 (2010)]*

*[Wineland et al, PRA **50**, 67 (1994)].*

General bound for phase estimation

Interferometer: $\hat{\rho} = \sum_l p_l |l\rangle$

$$\hat{\rho}_{out}(\theta) = e^{-i\theta J_{\vec{n}}} \hat{\rho} e^{i\theta J_{\vec{n}}}$$

$$\Delta\theta \geq \frac{1}{\sqrt{m}\sqrt{F_Q[\hat{\rho}, \hat{J}_n]}}$$

Cramer-Rao lower bound

$$F_Q[\hat{\rho}, \hat{J}_n] = \frac{1}{2} \sum_{l,k} \frac{(p_l - p_k)^2}{p_l + p_k} |\langle l|J_n|k\rangle|^2$$

quantum Fisher information

*[Braunstein et al, PRL **72**, 3439 (1994)]*

e.g, for pure states:

$$F_Q[\psi, \hat{J}_n] = 4\langle\psi|\Delta^2 \hat{J}_{\vec{n}}|\psi\rangle$$

$$\Delta\theta \langle \Delta \hat{J}_{\vec{n}} \rangle \geq \frac{1}{2\sqrt{m}}$$

(generalized uncertainty principle)

$$F_Q > N \Leftrightarrow \Delta\theta < \Delta\theta_{SN}$$

The quantum Fisher information captures the largest class of **useful states**

*[Braunstein et al, PRL **72**, 3439 (1994), Pezzé et al, PRL **102**, 100401 (2009)].*

squeezing does not: e.g. for $|\psi\rangle = (|\phi\rangle + |-\phi\rangle)/\sqrt{2}$ $\longrightarrow$ $\xi^2 > 1$

while $F_Q = N^2 \quad \Rightarrow \quad \Delta\theta = \frac{1}{N} \equiv \Delta\theta_{HL}$ (Heisenberg limit).

Multiparticle entanglement criteria

- ρ separable if it can be decomposed as $\rho = \sum_k P_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(N)}$
  non-separable (entangled) if not.

- $$\xi^2 < 1 \quad \Rightarrow \quad \hat{\rho} \text{ entangled.}$$
  $$\Rightarrow \quad \Delta\theta < \Delta\theta_{SN}$$

  *[Sorensen et al, Nature **409**, 63 (2001)].*

  *[Wineland et al, PRA **50**, 67 (1994)].*

- $$F_Q > N \quad \Rightarrow \quad \hat{\rho} \text{ entangled.}$$
  $$\Leftrightarrow \quad \Delta\theta < \Delta\theta_{SN}$$

  *[Pezzé et al, PRL **102**, 100401 (2009),*
  *Braunstein et al, PRL **72**, 3439 (1994)].*

Optimization over the directions $J_{\vec{n}}$ $\longrightarrow$ $F_Q = 4\mu_{max}\left[\Gamma_C\right]$

μ_{max} largest eigenvalue of the matrix $[\Gamma_C]_{i,j} = \frac{1}{2}\sum_{l,k} \frac{(p_l - p_k)^2}{p_l + p_k} \langle l|J_i|k\rangle\langle k|J_j|l\rangle$

[P. Hyllus et al, Phys. Rev. A 82, 012337 (2010)].

Dynamics of a quenched BJJ in the absence of noise

- Start with the coherent state $|\psi(t=0)\rangle = |\phi = 0\rangle$

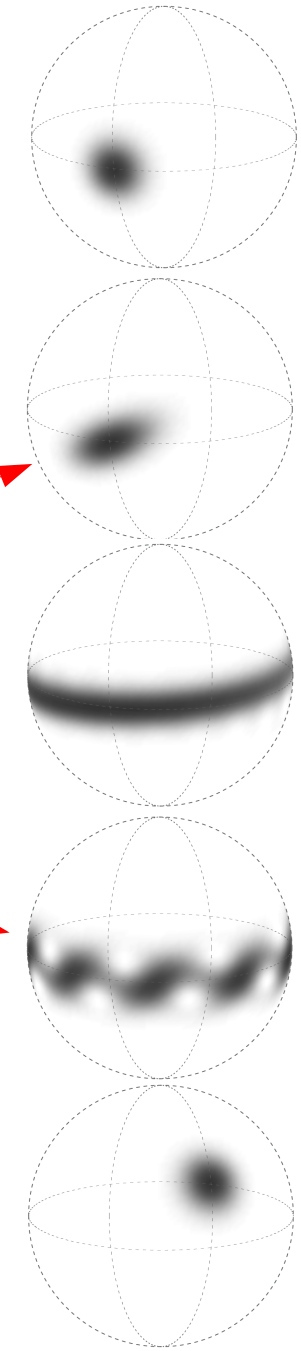
- Quench: set *K = 0* and let evolve in time with $\hat{H}^{(0)} = \chi \hat{J}_z^2$

- Short times: **squeezed state** (best squeezing $\xi_N^2 \approx \frac{1}{2}\left(\frac{3}{N}\right)^{\frac{2}{3}}$ at $t \sim 1/(\chi N^{\frac{2}{3}})$)*[Kitagawa et al, PRA **47**, 5138 (1993)].*

- $t_q \equiv \frac{\pi}{\chi q} \equiv \frac{T}{2q}$: q-component **macroscopic superposition**.
  *[Piazza et al, PRA **78**, 051601 (R) (2008); Ferrini et al, PRA **78**, 023606 (2008)].*

$$|\psi^{(0)}(t_q)\rangle = \sum_{k=0}^{q-1} c_k |\phi_k\rangle$$

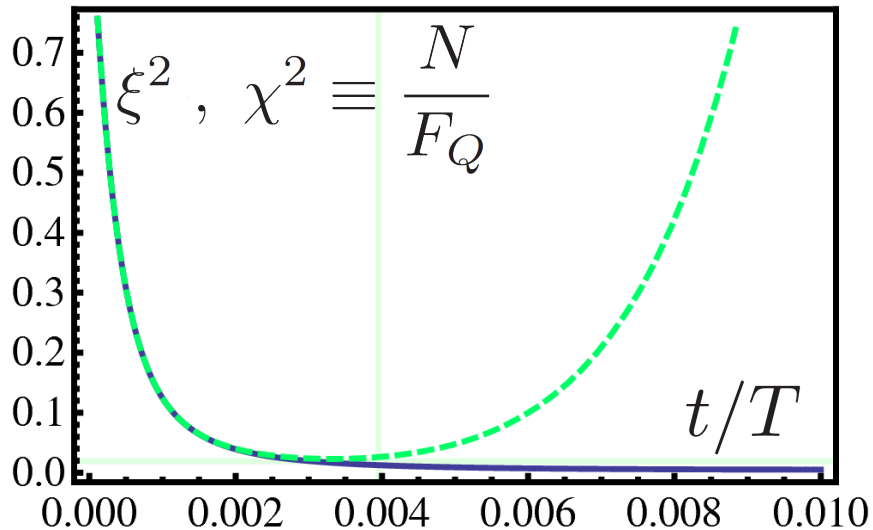
- *t = T/2:* reversed coherent state $|\psi(T/2)\rangle = |\phi = \pi\rangle$
  t = T: initial coherent state $|\psi(T)\rangle = |\phi = 0\rangle$.



[Figures from C.Gross PhD thesis, Heidelberg (2010)]

Spin squeezing during the quenched dynamics

In the absence of noise:

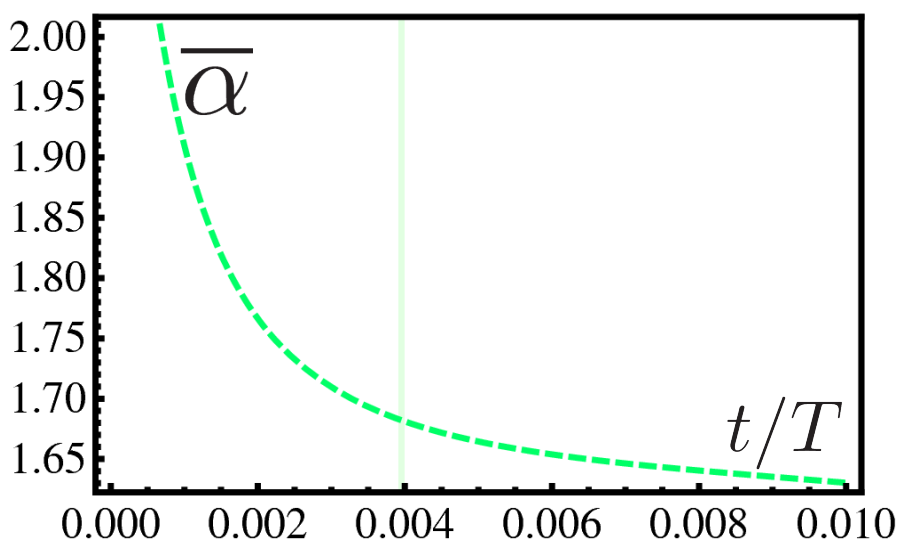


Optimum squeezing as a function of time

$$\xi^2 = \min_{\vec{n}} \frac{N\Delta^2 \hat{J}_{\vec{n}}}{\langle \hat{J}_{\vec{p_1}} \rangle^2 + \langle \hat{J}_{\vec{p_2}} \rangle^2}$$

$$t = t_{best} \sim 1/(\chi N^{\frac{2}{3}}) \Rightarrow \xi^2 \approx \frac{1}{2}\frac{3}{N}^{\frac{2}{3}}$$

*[Kitagawa et al, PRA **47**, 5138 (1993)].*



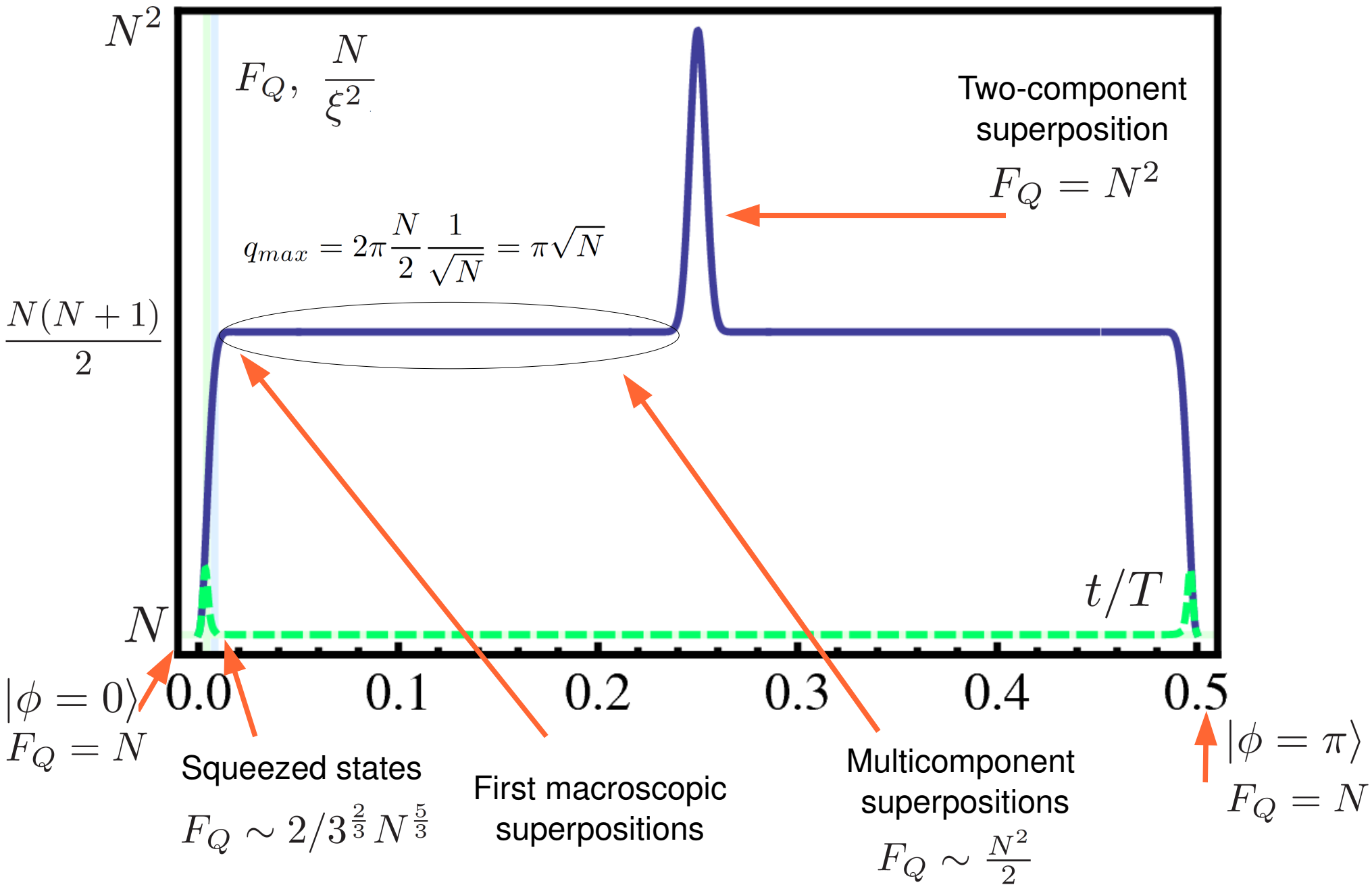
Angle of rotation of the optimizing direction

$$\hat{J}_{\vec{n}} = \cos\alpha \hat{J}_y + \sin\alpha \hat{J}_z$$

$$\overline{\alpha} = \frac{1}{2}\arctan\frac{\langle \hat{J}_y, \hat{J}_z \rangle}{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_z^2 \rangle} + k\frac{\pi}{2}$$

$$N = 400; \quad \chi = 0.13\pi\mathrm{Hz}$$

Fisher information during the quenched dynamics

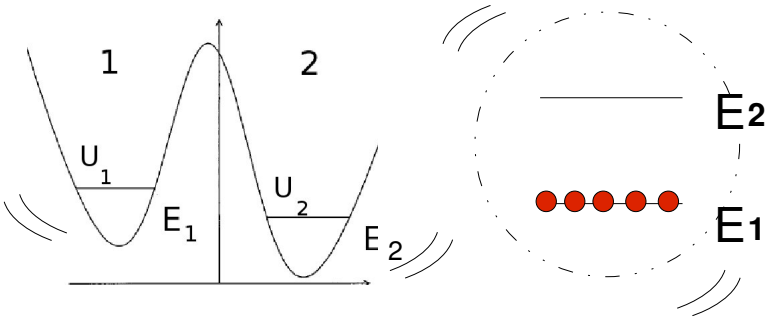


Quenched dynamics in the presence of phase noise

$$\hat{H} = \chi \hat{J}_z^2 - \lambda(t)\hat{J}_z$$

$$\lambda = E_1 - E_2$$

randomly fluctuating



At each realization

$$|\psi(t)\rangle = e^{-i\int_0^t d\tau \lambda(\tau)\hat{J}_z}|\psi^{(0)}(t)\rangle \equiv e^{-i\phi\hat{J}_z}|\psi^{(0)}(t)\rangle$$

Rotation of the quantum state

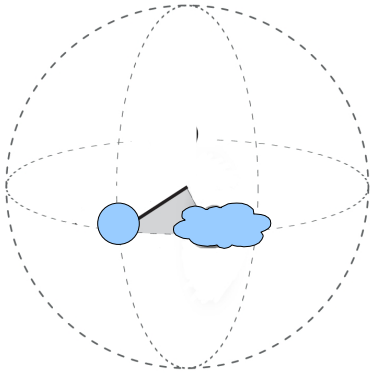
Average over the realizations

$$\hat{\rho}(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} = \int_0^t d\phi f(t,\phi) e^{-i\phi\hat{J}_z}\hat{\rho}^{(0)}(t)e^{i\phi\hat{J}_z}$$

Rotation + spreading

$$f(t,\phi) = \frac{e^{-(\phi+\overline{\lambda}t)^2}}{\sqrt{2\pi a^2(t)}} \qquad a^2(t) = \int_0^t d\tau \int_0^t d\tau' \left(\overline{\lambda(\tau)\lambda(\tau')} - \overline{\lambda}^2\right)$$

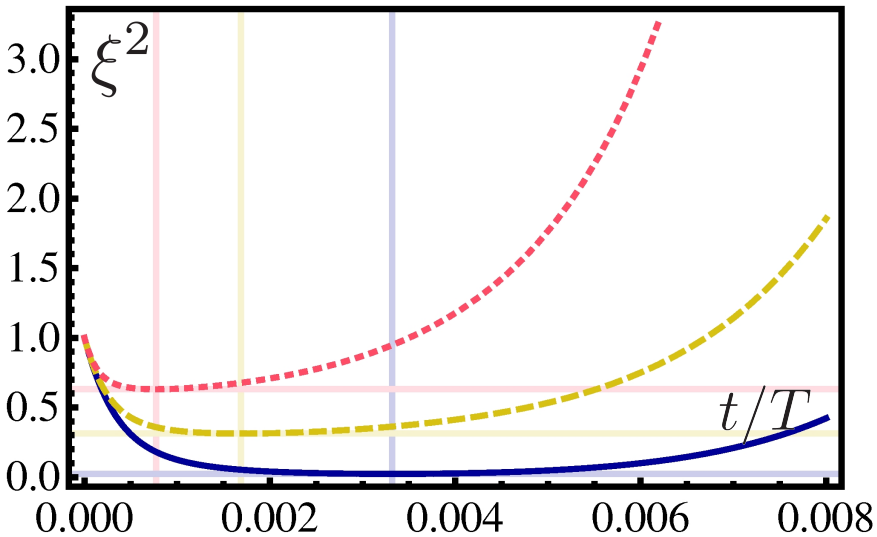
$$\langle n|\hat{\rho}(t)|n'\rangle = e^{-\frac{a^2(t)(n-n')^2}{2}} e^{i\overline{\lambda}t(n-n')}\langle n|\hat{\rho}^{(0)}(t)|n'\rangle$$

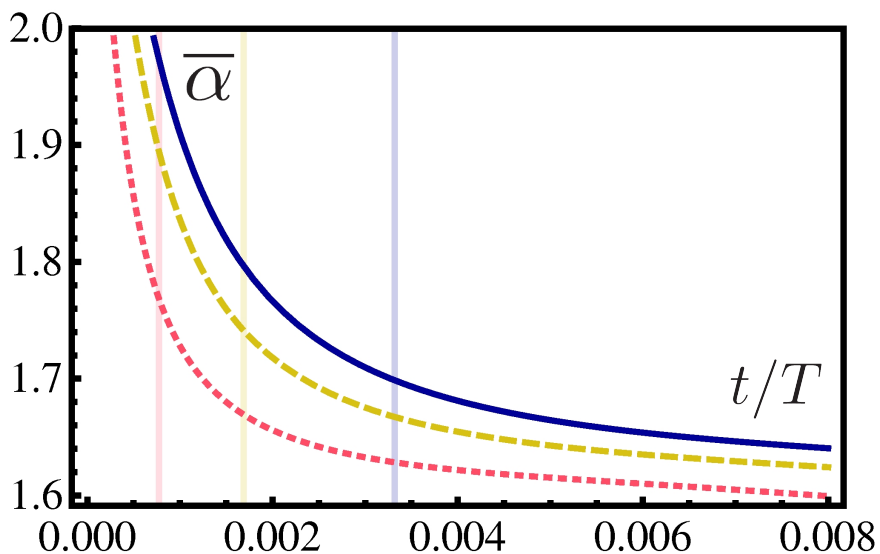


[G.Ferrini et al, arXiv:0911.0655 (2009)]

Effect of phase noise on spin squeezing

$$\langle n|\hat{\rho}(t)|n'\rangle = e^{-\frac{a^2(t)(n-n')^2}{2}} e^{i\overline{\lambda}t(n-n')} \langle n|\hat{\rho}^{(0)}(t)|n'\rangle$$





$N = 400; \quad \chi = 0.13\pi\text{Hz}$

Short time approximation: $t \ll t_c$

$$a^2(t) = \int_0^t d\tau \int_0^t d\tau' \left(\overline{\lambda(\tau)\lambda(\tau')} - \overline{\lambda}^2\right) \sim \Delta^2\lambda\, t^2$$

Optimum squeezing as a function of time

$$\xi^2 = \min_{\vec{n}} \frac{N\Delta^2 \hat{J}_{\vec{n}}}{\langle \hat{J}_{\vec{p_1}}\rangle^2 + \langle \hat{J}_{\vec{p_2}}\rangle^2}$$

$$\hat{J}_{\vec{n}} = \cos\alpha \hat{J}_y + \sin\alpha \hat{J}_z$$

Angle of rotation of the optimum direction

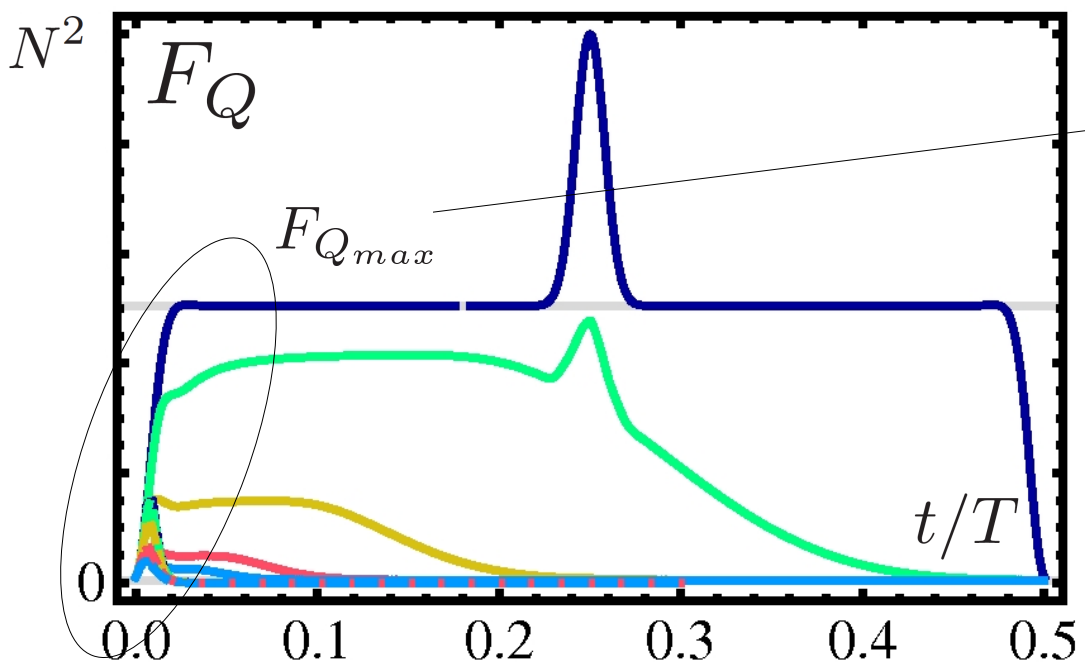
$$\overline{\alpha} = \frac{1}{2}\arctan \frac{\text{Tr}\left[\hat{\rho}\left\{\hat{J}_y, \hat{J}_z\right\}\right]}{\text{Tr}\left[\hat{\rho}\hat{J}_y^2\right] - \text{Tr}\left[\hat{\rho}\hat{J}_z^2\right]} + k\frac{\pi}{2}$$

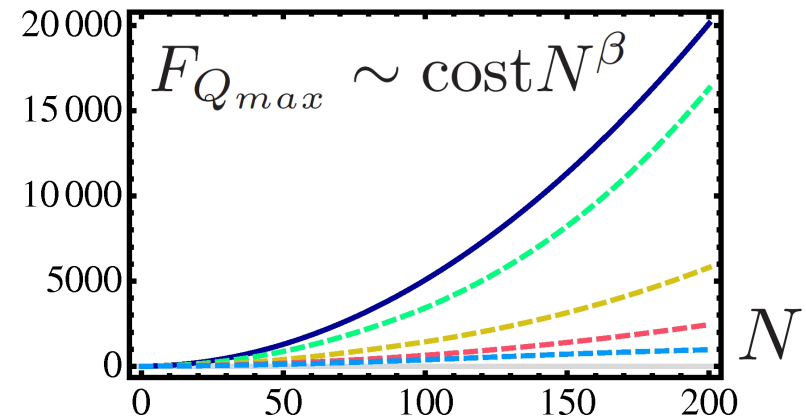
The presence of noise changes the optimum time for squeezing and the angle of rotation.

$\Delta\lambda = 0, 5, 10 \text{ Hz} \qquad \overline{\lambda} = 0$

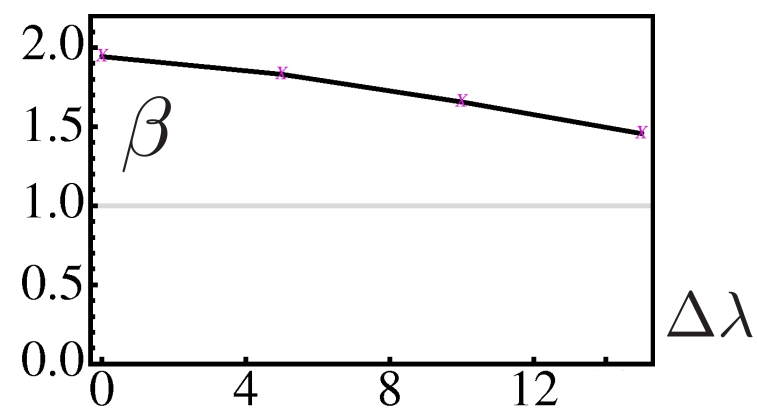
Effect of phase noise on Fisher information

Most useful quantum state = maximum of the Fisher information

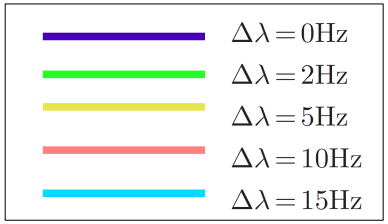




$$\mathrm{Log} F_{Q_{max}} \propto \beta \mathrm{Log} N$$

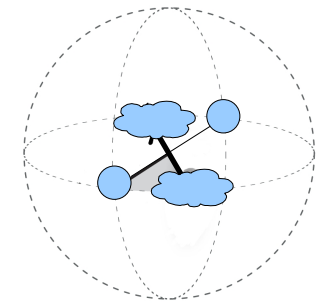


The scaling of the local maximum of the Fisher information with *N* stays better than *1* at intermediate noise strength $0 < \Delta\lambda \lesssim 10\mathrm{Hz} \quad (\chi = \pi\mathrm{Hz})$



Effect of noise on macroscopic superpositions $|\psi^{(0)}(t_q)\rangle = \sum_{k=0}^{q-1} c_k |\phi_k\rangle$

$$\langle n|\hat{\rho}(t)|n'\rangle = e^{-\frac{a^2(t)(n-n')^2}{2}} e^{i\overline{\lambda}t(n-n')} \langle n|\hat{\rho}^{(0)}(t)|n'\rangle$$

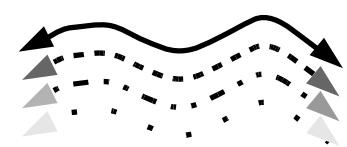


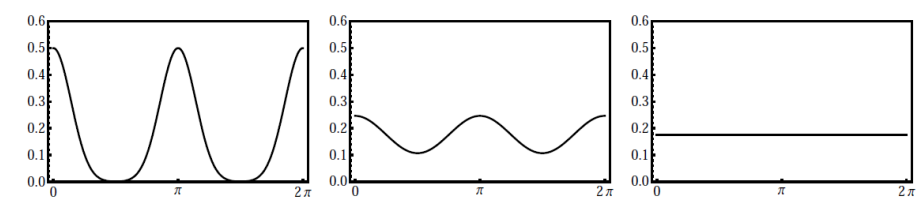
$$\hat{\rho}^{(0)}(t_q) = \sum_{k=0}^{q-1} |c_k|^2 |\phi_k\rangle\langle\phi_k| + \sum_{k\neq k'0}^{q-1} c_k c_k'^* |\phi_k\rangle\langle\phi_k'|$$

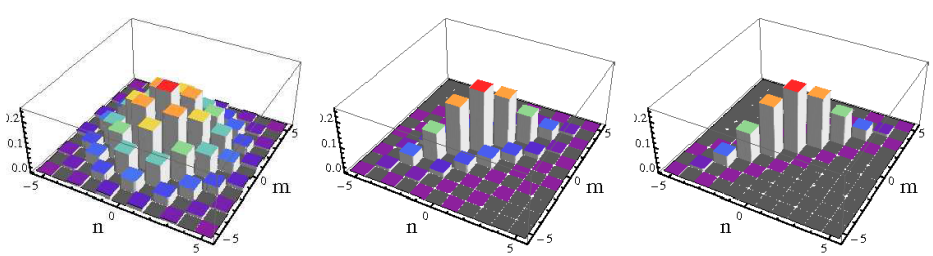
$$\hat{\rho}_\infty = \frac{1}{2\pi}\int_0^{2\pi} d\phi |\phi\rangle\langle\phi| = \frac{1}{2^N}\binom{N}{\frac{N}{2}+n} |n\rangle\langle n|$$

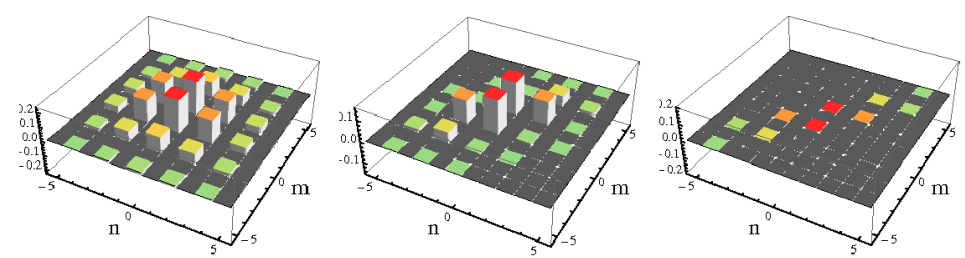
0

$$\overline{\lambda} = 0$$







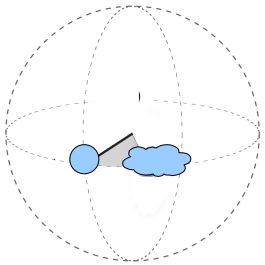


Phase relaxation → SAME NOISE STRENGTH! ← *Decoherence*

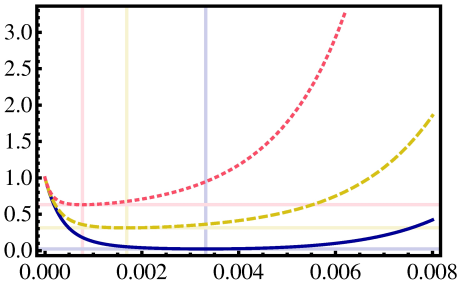
Independent on *N* if $U_1 = U_2$ *[G.Ferrini et al, arXiv:0911.0655]*

Conclusions

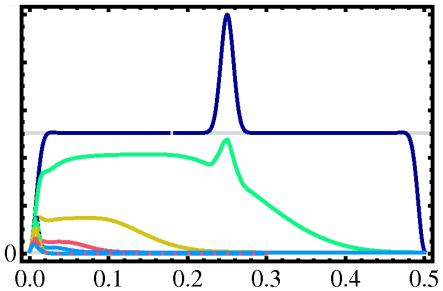
- Exactly solvable model for phase noise in a Bose Josephson junction which undergoes a quenched dynamics.



- The presence of phase noise degrades the squeezing and changes both its optimum time of formation and the orientation.



- At intermediate noise strength, quantum correlations useful for interferometry survive. At each *N* and noise strength, we can determine the most useful quantum state.



- Decoherence of macroscopic superpositions occur with a noise rate which does not depend on *N,* and with the same scale as relaxation.