Useful quantum states in the presence of noise in a Bose Josephson junction

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Outline

- Bosonic Josephson junctions
- Useful entangled states for atom interferometry
- Quenched unitary dynamics
  - Squeezed states;
  - Macroscopic superpositions;
- Quenched dynamics in the presence of noise
  - Effect on squeezed states?
  - Effect on macroscopic superpositions?
  - Most useful states for interferometry?
Experimental realizations of a Bose Josephson junction (BJJ)

\[ \hat{H}^{(0)} = E_1 \hat{a}_1^\dagger \hat{a}_1 + E_2 \hat{a}_2^\dagger \hat{a}_2 - K (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + U_1 \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + U_2 \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \]

\[ \chi = U_1 + U_2 \]

\[ \lambda = (E_1 - E_2) + (N - 1) \frac{U_1 - U_2}{2} \]

\[ \chi = U_1 + U_2 - 2U_{12} \]

\[ \hat{J}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \equiv \hat{n} \]

\[ \hat{J}_y = -i \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \]

\[ \hat{J}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \]

\[ N = a_1^\dagger a_1 + a_2^\dagger a_2 \]

\[ J^2 = N/2 (N/2 + 1) \]

\[ \hat{H}^{(0)} = \chi \hat{J}_z^2 - \lambda \hat{J}_z - 2K \hat{J}_x \]
Ground state/possible basis of a Bose Josephson junction

\[ \hat{H}^{(0)} = \chi \hat{J}_z^2 - \lambda \hat{J}_z - 2K \hat{J}_x = \chi (\hat{J}_z - \frac{\lambda}{2\chi})^2 - 2K \hat{J}_x \]

\[ \chi \gg KN \] (Fock regime)

\[ |\psi_{GS}\rangle = |n\rangle \]

Definite atom number in each of the two modes

\[ \hat{J}_z |n\rangle = n |n\rangle \quad n = \text{Int} \left[ \frac{\lambda}{2\chi} \right] \]

| Fock state |

\[ \chi N \ll K \] (Rabi regime)

\[ |\psi_{GS}\rangle = |\phi\rangle \]

Atomic phase state (equator of the Bloch sphere)

\[ |\phi\rangle = \frac{1}{2^{N/2}} \sum_{n=-N/2}^{N/2} \left( \frac{N}{2} + n \right)^{1/2} e^{-i\phi \left( \frac{N}{2} + n \right)} |n\rangle \]

\[ = \frac{(a_1^\dagger + e^{-i\phi} a_2^\dagger)^N}{\sqrt{N!}} |0\rangle \]

All the atoms are in a coherent superposition of the two modes
Ramsey atom-interferometer

Goal: to estimate a phase shift $\Theta$ with the highest possible precision.

$$|\psi\rangle_{out} = e^{-iJ_y \frac{\pi}{2}} e^{iJ_z \theta} e^{iJ_y \frac{\pi}{2}} |n = -N/2\rangle$$

$$= e^{i\theta J_x} |n = -N/2\rangle$$

$$\langle \hat{J}_z \rangle_{out} = \frac{N}{2} \cos \theta$$

$$\Delta \theta = \frac{\Delta \hat{J}_{out}}{|\partial \langle \hat{J}_{out} \rangle / \partial \theta|}$$

$$\Delta \hat{J}_z = \sqrt{N}/2$$

$$\max |\partial \langle \hat{J}_{out} \rangle / \partial \theta| = N/2$$

To realize the ports:

$\rightarrow$ Set $\chi \sim 0$;

$$\hat{H}^{(0)} = -\lambda \hat{J}_z - 2|K| (\cos \delta \hat{J}_x + \sin \delta \hat{J}_y)$$

$\rightarrow$ Tune $K$, $\lambda$, $\delta$ and let evolve in time.

Enhanced phase resolution by squeezing

\[ \Delta \hat{J}_{\tilde{n}} \Delta \hat{J}_{\tilde{p}_1} \geq \frac{|\langle \hat{J}_{\tilde{p}_2} \rangle|}{2} \]

\[ \Delta \hat{J}_{\tilde{n}} < \Delta \hat{J}_{\tilde{p}_1} \]

\[ |\Delta \theta| = \frac{\Delta \hat{J}_{\text{zout}}}{\frac{\partial \langle \hat{J}_{\text{zout}} \rangle}{\partial \theta}} \]

\[ \xi^2 = \min_{\tilde{n}} \frac{N \Delta^2 \hat{J}_{\tilde{n}}}{\langle \hat{J}_{\tilde{p}_1} \rangle^2 + \langle \hat{J}_{\tilde{p}_2} \rangle^2} \]

\[ \xi^2 < 1 \implies \Delta \theta < \Delta \theta_{SN} \equiv \frac{1}{\sqrt{N}} \]

[C. Gross et al, Nature 464, 1165 (2010)]

[Wineland et al, PRA 50, 67 (1994)].
General bound for phase estimation

Interferometer: \[ \hat{\rho} = \sum_l p_l |l\rangle \]

\[ \hat{\rho}_{out}(\theta) = e^{-i\theta J_n} \hat{\rho} e^{i\theta J_n} \]

\[ F_Q[\hat{\rho}, \hat{J}_n] = \frac{1}{2} \sum_{l,k} \frac{(p_l - p_k)^2}{p_l + p_k} |\langle l | J_n | k \rangle|^2 \]

Cramer-Rao lower bound

\[ \Delta \theta \geq \frac{1}{\sqrt{m \sqrt{F_Q[\hat{\rho}, \hat{J}_n]}}} \]

quantum Fisher information

[Braunstein et al, PRL 72, 3439 (1994)]

e.g., for pure states: \[ F_Q[\psi, \hat{J}_n] = 4 \langle \psi | \Delta^2 \hat{J}_n | \psi \rangle \]

\[ \Delta \theta \langle \Delta \hat{J}_n \rangle \geq \frac{1}{2 \sqrt{m}} \] (generalized uncertainty principle)

The quantum Fisher information captures the largest class of useful states

[Braunstein et al, PRL 72, 3439 (1994), Pezzé et al, PRL 102, 100401 (2009)].

Squeezing does not: e.g., for \[ |\psi\rangle = (|\phi\rangle + |-\phi\rangle)/\sqrt{2} \] \[ \xi^2 > 1 \]

while \[ F_Q = N^2 \Rightarrow \Delta \theta = \frac{1}{N} \equiv \Delta \theta_{HL} \] (Heisenberg limit).
Multiparticle entanglement criteria

- $\rho$ separable if it can be decomposed as $\rho = \sum_k P_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(N)}$
- $\xi^2 < 1 \implies \hat{\rho}$ entangled.
- $F_Q > N \implies \hat{\rho}$ entangled.

$\xi^2 < 1 \implies \Delta \theta < \Delta \theta_{SN}$

$F_Q = 4 \mu_{max} [\Gamma_C]$ 

$\mu_{max}$ largest eigenvalue of the matrix $[\Gamma_C]_{i,j} = \frac{1}{2} \sum_{l,k} \frac{(p_l - p_k)^2}{p_l + p_k} \langle l | J_i | k \rangle \langle k | J_j | l \rangle$


[Sorensen et al, Nature 409, 63 (2001)].

[Wineland et al, PRA 50, 67 (1994)].

[Pezzé et al, PRL 102, 100401 (2009), Braunstein et al, PRL 72, 3439 (1994)].
Dynamics of a quenched BJJ in the absence of noise

- Start with the coherent state $|\psi(t = 0)\rangle = |\phi = 0\rangle$

- Quench: set $K = 0$ and let evolve in time with $\hat{H}^{(0)} = \chi \hat{J}_z^2$

- Short times: **squeezed state** (best squeezing at $t \sim 1/(\chi N^{2/3})$) [Kitagawa et al, PRA 47, 5138 (1993)].

- $t_q \equiv \frac{\pi}{\chi q} \equiv \frac{T}{2q}$: q-component **macroscopic superposition**. [Piazza et al, PRA 78, 051601 (R) (2008); Ferrini et al, PRA 78, 023606 (2008)].

- $t = T/2$: reversed coherent state $|\psi(T/2)\rangle = |\phi = \pi\rangle$
- $t = T$: initial coherent state $|\psi(T)\rangle = |\phi = 0\rangle$.

[Figures from C.Gross PhD thesis, Heidelberg (2010)]
Spin squeezing during the quenched dynamics

In the absence of noise:

Optimum squeezing as a function of time

\[ \xi^2 = \min_{\vec{n}} \frac{N \Delta^2 \hat{J}_{\vec{n}}}{\langle \hat{J}_{\vec{p}_1}^2 \rangle + \langle \hat{J}_{\vec{p}_2}^2 \rangle} \]

\[ t = t_{\text{best}} \sim \frac{1}{(\chi N^{\frac{2}{3}})} \quad \Rightarrow \quad \xi^2 \approx \frac{1}{2} \frac{3^{\frac{2}{3}}}{N} \]

[Kitagawa et al, PRA 47, 5138 (1993)].

Angle of rotation of the optimizing direction

\[ \hat{J}_{\vec{n}} = \cos \alpha \hat{J}_y + \sin \alpha \hat{J}_z \]

\[ \overline{\alpha} = \frac{1}{2} \arctan \frac{\langle \hat{J}_y, \hat{J}_z \rangle}{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_z^2 \rangle} + k \frac{\pi}{2} \]

\[ N = 400; \quad \chi = 0.13\pi \text{Hz} \]
Fisher information during the quenched dynamics

\[ F_Q, \quad \frac{N}{\xi^2} \]

\[ q_{\text{max}} = 2\pi \frac{N}{2} \frac{1}{\sqrt{N}} = \pi \sqrt{N} \]

Two-component superposition
\[ F_Q = N^2 \]

First macroscopic superpositions
\[ F_Q \sim 2/3^{2/3} N^{5/3} \]

Multicomponent superpositions
\[ F_Q \sim \frac{N^2}{2} \]
Quenched dynamics in the presence of phase noise

\[ \hat{H} = \chi \hat{J}_z^2 - \lambda(t) \hat{J}_z \]

\[ \lambda = E_1 - E_2 \]

randomly fluctuating

At each realization

\[ |\psi(t)\rangle = e^{-i \int_0^t d\tau \lambda(\tau) \hat{J}_z} |\psi(0)(t)\rangle \equiv e^{-i \phi \hat{J}_z} |\psi(0)(t)\rangle \]

Average over the realizations

\[ \hat{\rho}(t) = \frac{|\psi(t)\rangle \langle \psi(t)|}{\langle \psi(t) | \psi(t) \rangle} = \int_0^t d\phi f(t, \phi) e^{-i \phi \hat{J}_z} \hat{\rho}(0)(t) e^{i \phi \hat{J}_z} \]

\[ f(t, \phi) = \frac{e^{-(\phi + \bar{\lambda} t)^2}}{\sqrt{2\pi a^2(t)}} \]

\[ a^2(t) = \int_0^t d\tau \int_0^t d\tau' \left( \frac{\bar{\lambda}(\tau) \bar{\lambda}(\tau')}{2} - \bar{\lambda}^2 \right) \]

\[ \langle n | \hat{\rho}(t) | n' \rangle = e^{-\frac{a^2(t) (n-n')^2}{2}} e^{i \bar{\lambda} t (n-n')} \langle n | \hat{\rho}(0)(t) | n' \rangle \]

Effect of phase noise on spin squeezing

\[ \langle n | \hat{\rho}(t) | n' \rangle = e^{-\frac{a^2(t)(n-n')^2}{2}} e^{i\lambda t(n-n')} \langle n | \hat{\rho}^{(0)}(t) | n' \rangle \]

Short time approximation: \( t \ll t_c \)

\[ a^2(t) = \int_0^t d\tau \int_0^t d\tau' \left( \frac{\lambda(\tau)\lambda(\tau')}{2} - \overline{\lambda}^2 \right) \sim \Delta^2 \lambda t^2 \]

Optimum squeezing as a function of time

\[ \xi^2 = \min_{\bar{n}} \frac{N \Delta^2 \hat{J}_{\bar{n}}}{\langle \hat{J}_{p_1} \rangle^2 + \langle \hat{J}_{p_2} \rangle^2} \]

\[ \hat{J}_{\bar{n}} = \cos \alpha \hat{J}_y + \sin \alpha \hat{J}_z \]

Angle of rotation of the optimum direction

\[ \overline{\alpha} = \frac{1}{2} \arctan \frac{\text{Tr} \left[ \hat{\rho} \left\{ \hat{J}_y, \hat{J}_z \right\} \right]}{\text{Tr} \left[ \hat{\rho} \hat{J}_y^2 \right] - \text{Tr} \left[ \hat{\rho} \hat{J}_z^2 \right]} + k \frac{\pi}{2} \]

The presence of noise changes the optimum time for squeezing and the angle of rotation.

\[ \Delta \lambda = 0, 5, 10 \text{ Hz} \quad \overline{\lambda} = 0 \]
Effect of phase noise on Fisher information

Most useful quantum state = maximum of the Fisher information

The scaling of the local maximum of the Fisher information with $N$ stays better than 1 at intermediate noise strength $0 < \Delta \lambda \lesssim 10\text{Hz}$ ($\chi = \pi\text{Hz}$)
Effect of noise on macroscopic superpositions

\[ \langle n | \hat{\rho}(t) | n' \rangle = e^{-\frac{a^2(t)(n-n')^2}{2}} e^{i\tilde{\lambda}t(n-n')} \langle n | \hat{\rho}^{(0)}(t) | n' \rangle \]

\[ \hat{\rho}^{(0)}(t) = \sum_{k=0}^{q-1} c_k |\phi_k\rangle \langle \phi_k| \]

\[ \hat{\rho}_\infty = \frac{1}{2\pi} \int_0^{2\pi} d\phi |\phi\rangle \langle \phi| = \frac{1}{2N} \left( \frac{N}{N/2+n} \right) |n\rangle \langle n| \]

Phase relaxation \quad \text{SAME NOISE STRENGTH!} \quad \text{Decoherence}

Independent on \( N \) if \( U_1 = U_2 \) [G.Ferrini et al, arXiv:0911.0655]
Conclusions

- Exactly solvable model for phase noise in a Bose Josephson junction which undergoes a quenched dynamics.

- The presence of phase noise degrades the squeezing and changes both its optimum time of formation and the orientation.

- At intermediate noise strength, quantum correlations useful for interferometry survive. At each $N$ and noise strength, we can determine the most useful quantum state.

- Decoherence of macroscopic superpositions occur with a noise rate which does not depend on $N$, and with the same scale as relaxation.