Phase spreading of a Bose-Einstein condensate at nonzero temperature

Emilia Witkowska[1], Alice Sinatra[2], Yvan Castin[2]

[1] Institute of Physics, Polish Academy of Sciences, Warsaw, Poland

[2] Laboratoire Kastler Brossel, Ecole Normale Supérieure, Paris, France

Considered system

- **the Bose condensed gas at thermal equilibrium** — statistical ensembles (microcanonical or canonical) of Bose c-fields ψ
- **3D spatially homogeneous system with periodic boundary conditions** — box of volume V — dynamics of the Bose fields given by the nonlinear Schrödinger equation (NLSE)
- small non condensed fraction — **low temperature**
- total number of atoms fixed to N

Condensate

Special case — box potential $V(\mathbf{r}) = 0$ **with periodic boundary condition.** Natural basis of the problem are plane waves:

$$\psi(\mathbf{r}, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}$$

- **the condensate** is the zero momentum component

$$a_0(t) = e^{i\theta(t)} \sqrt{N_0(t)}$$

- and $\theta(t)$ is **the condensate phase**

What we expect?

$T = 0$**:**
correlation function $\langle a_0^*(t) a_0(0)\rangle$ oscillates with frequency μ —
no phase spreading (Beliaev 1958)

What we expect?

$T = 0$**:**
correlation function $\langle a_0^*(t) a_0(0) \rangle$ oscillates with frequency μ —
no phase spreading (Beliaev 1958)

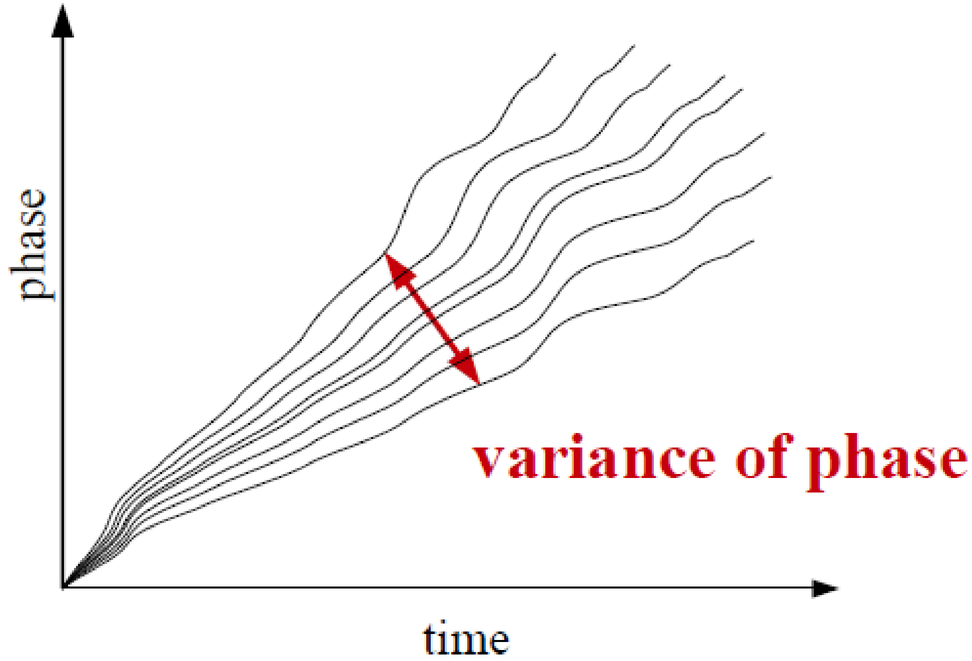
$T \neq 0$**:**
interactions with **thermal atoms perturb the condensate phase** —
correlation function $\langle a_0^*(t) a_0(0) \rangle$ decays at long time

What we expect?

$T = 0$**:**
correlation function $\langle a_0^*(t)a_0(0)\rangle$ oscillates with frequency μ —
no phase spreading (Beliaev 1958)

$T \neq 0$**:**
interactions with **thermal atoms perturb the condensate phase** —
correlation function $\langle a_0^*(t)a_0(0)\rangle$ decays at long time

Due to interactions condensate phase will be not exactly the same in each realisation of the field. Phase spreads in time.



Previous predictions at $T \neq 0$

- **Quantum optics inspired approaches:** **Var** $\varphi(t) \sim 2Dt$.
  D. Jaksch, C. Gardiner, K. Gheri, P. Zoller PRA (1998).
  R. Graham PRL (1998), PRA, JMO (2000).

- **Manybody approach :** **Var** $\varphi(t) \sim \alpha_{Bog} t^2$
  A. Kuklov, J. Birman PRA (1998). The prediction neglects interactions among Bogoliubov modes.

Previous predictions at $T \neq 0$

- **Quantum optics inspired approaches:** **Var** $\varphi(t) \sim 2Dt$.
  D. Jaksch, C. Gardiner, K. Gheri, P. Zoller PRA (1998).
  R. Graham PRL (1998), PRA, JMO (2000).

- **Manybody approach :** **Var** $\varphi(t) \sim \alpha_{Bog} t^2$
  A. Kuklov, J. Birman PRA (1998). The prediction neglects interactions among Bogoliubov modes.

Important aspects not included in previous treatments:

- **Interactions** among the non-condensed Bogoliubov modes
- condensate and non-condensate parts are **isolated system**

Numerical experiment

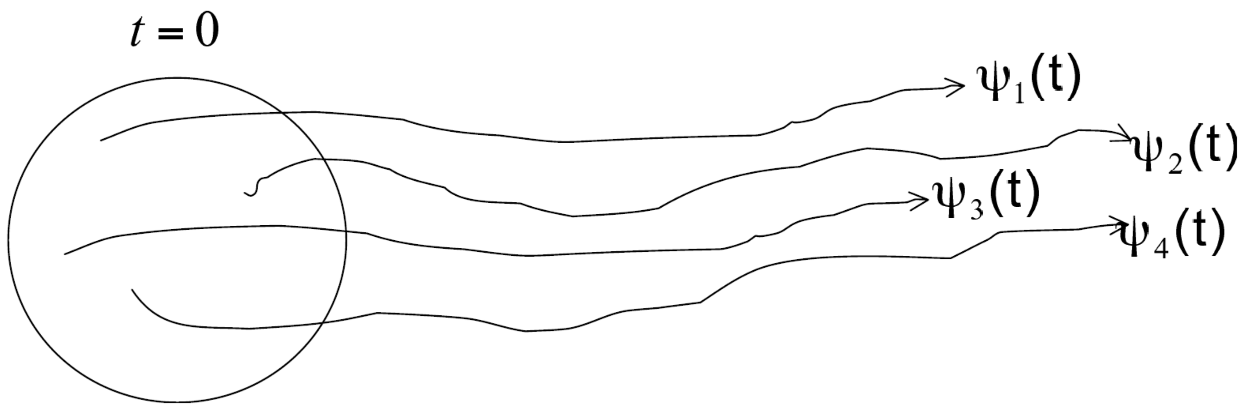
The condensate and the thermal fraction are describing by single c-filed **(the classical fields method)**

- **advantages:** classical field can be simulated exactly and contain the full non-linear dynamics
- **UV catastrophe:** $\bar{n}_{\mathbf{k}} = \frac{k_B T}{\epsilon_{\mathbf{k}}} \neq \frac{1}{e^{\beta \epsilon_{\mathbf{k}}} - 1}$ cut-off is needed (results depend on cut-off)

A. Sinatra *et al*, PRL **87**, 210404 (2001); M.J. Davis *et al*, PRL **87**, 160402 (2001); M. Brewczyk *et al*, J.Phys.B **40**, R1 (2007)

Classical fields simulations

- at $t = 0$ and temperature T **generate** ensemble of fields $\psi_j(\mathbf{r}, t)$



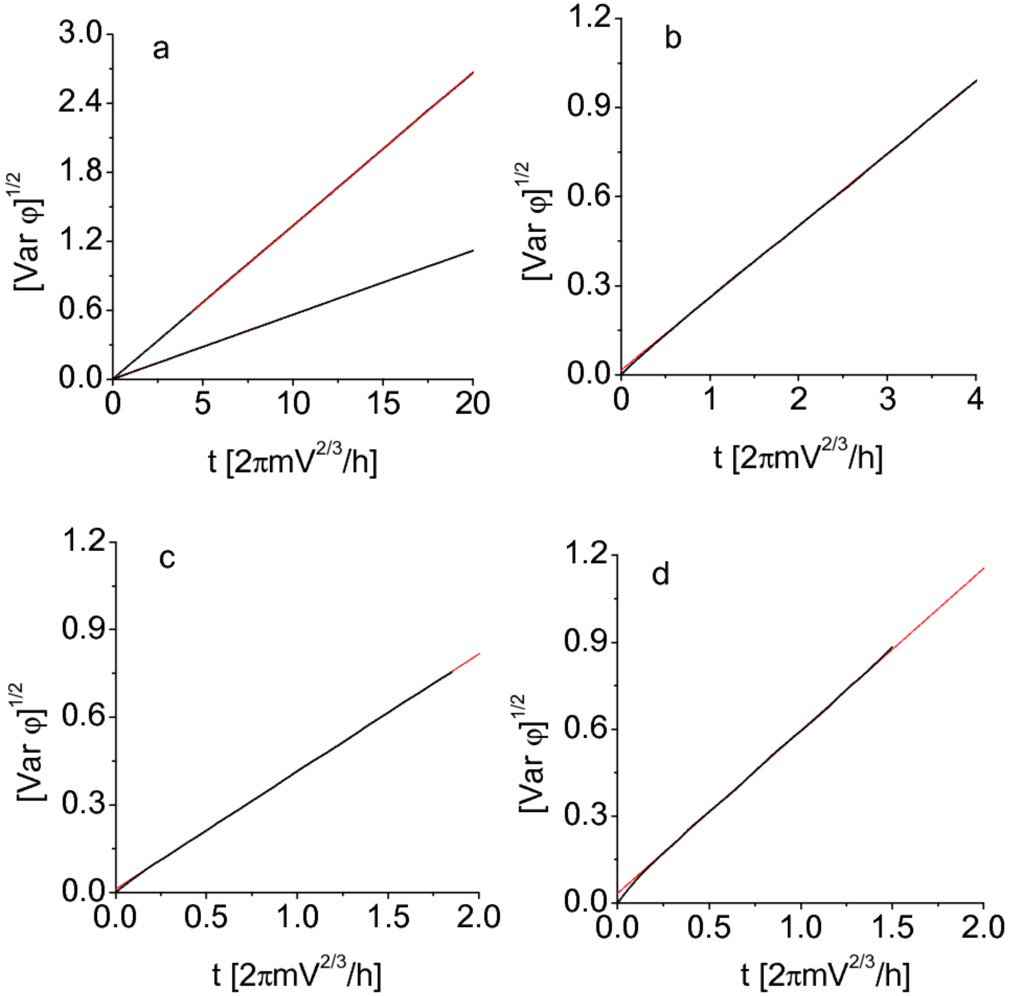
- **evolve** with NLSE:

$$i\hbar\dot{\psi}_j(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2m}\Delta + g|\psi_j(\mathbf{r}, t)|^2\right)\psi_j(\mathbf{r}, t)$$

- **extract the condensate** mode $a_0(t) = |a_0|e^{-i\theta(t)}$ by using FFT
- **calculate** variance of the phase $\mathrm{Var}\,\varphi(t) = \langle\varphi(t)^2\rangle - \langle\varphi(t)\rangle^2$ with $\varphi(t) = \theta(t) - \theta(0)$, where averages are over fields' realisations

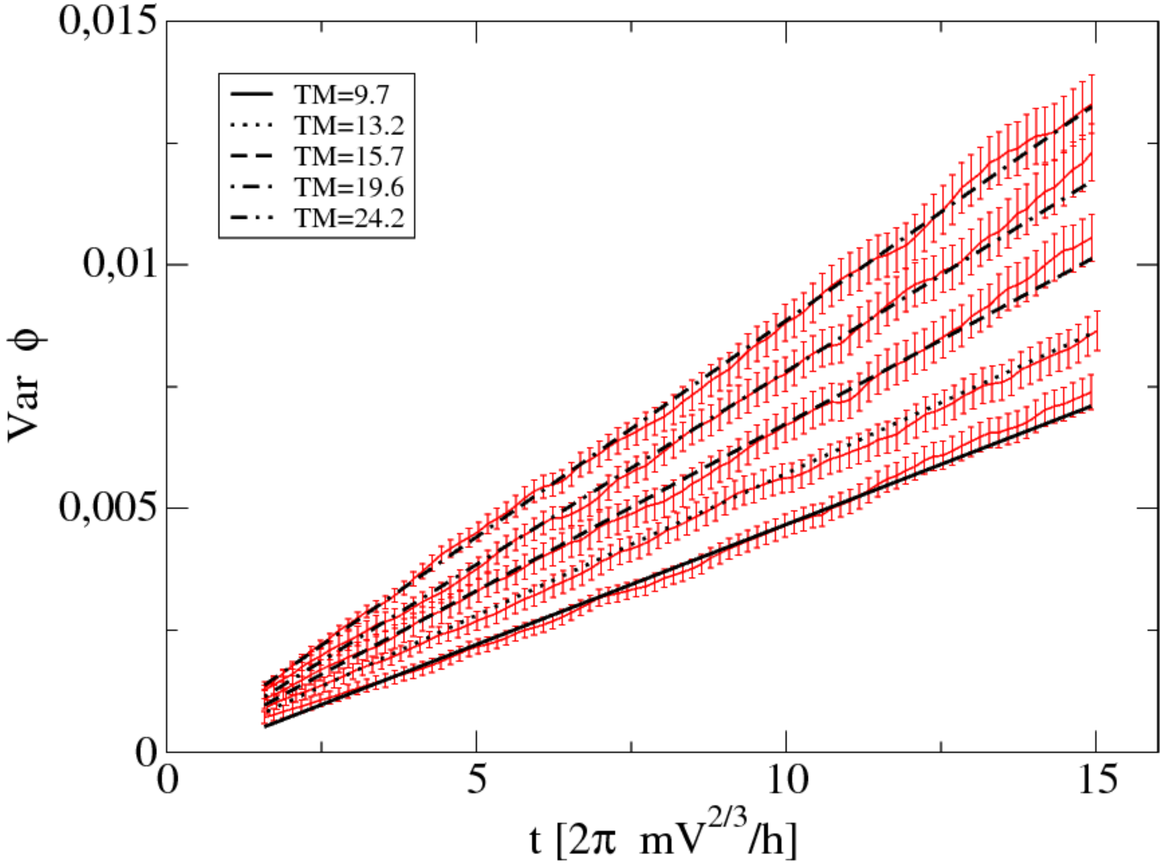
Initial state in canonical ensemble

Classical fields generated for a given temperature T and number of atoms N. **Fluctuations of energy are present** in the field samples.



Initial state in microcanonical ensemble

Classical fields generated for **given energy** E and **number of atoms** N.



For convenience, we parametrize the microcanonical ensemble by the temperature T **such that the mean energy in the canonical ensemble at temperature** T **is equal to** E**.**

Result

Variance of the phase depends strongly on the ensemble!

In the canonical: ballistic expansion $\operatorname{Var}\varphi \sim t^2$.

In the microcanonical: diffusive motion $\operatorname{Var}\varphi \sim t$.

Analytical approach

Split the c-field ψ into condensate and non condensed parts:

$$\psi(\mathbf{r},t) = \frac{a_0(t)}{\sqrt{V}} + \psi_\perp(\mathbf{r},t)$$

- **the non-condensed part** expanded over the Bogoliubov modes:

$$\psi_\perp(\mathbf{r},t) = e^{-i\theta}\sum_{\mathbf{k}\neq 0}(U_\mathbf{k} b_\mathbf{k}(t) + V_\mathbf{k} b^*_{-\mathbf{k}}(t))\frac{e^{i\mathbf{kr}}}{\sqrt{V}}$$

- **Hamiltonian:**

$$H = E_0 + \sum_{\mathbf{k}\neq 0}\epsilon_\mathbf{k} b^*_\mathbf{k} b_\mathbf{k} \; + \; \textbf{cubic terms} \; + \; \textbf{quartic terms}$$

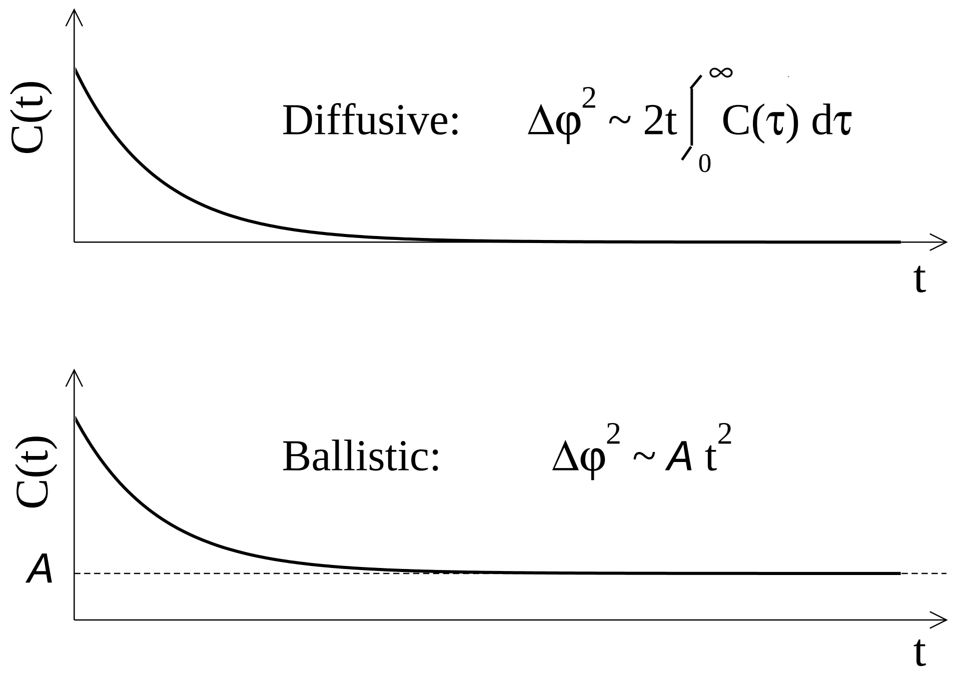
- **condensate mode:** $a_0(t) = e^{i\theta(t)}\sqrt{N_0(t)}$
- **occupation number of Bogoliubov modes:** $n_\mathbf{k}(t) = \langle b(t)^*_\mathbf{k} b(t)_\mathbf{k}\rangle$

Correlation function

phase variance:

$$\mathrm{Var}\,\varphi(t) = 2t\int_0^t C(\tau)d\tau - 2\int_0^t \tau C(\tau)d\tau$$

where **correlation function:** $C(t) = \langle\dot\varphi(t)\dot\varphi(0)\rangle - \langle\dot\varphi(t)\rangle\langle\dot\varphi(0)\rangle$



How to calculate $C(t)$?

One get **phase derivative** from equations $i\hbar\dot{a}_0 = \frac{\partial H}{\partial a_0^*}$ and $\dot{\varphi} = -Re\left[\frac{i\dot{a}_0}{a_0}\right]$:

$$\dot{\varphi} \simeq -\frac{\mu}{\hbar} + \sum_{\mathbf{k}\neq 0} A_{\mathbf{k}} n_{\mathbf{k}} ,$$

$$C(t) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} A_{\mathbf{k}} A_{\mathbf{k}'} \langle \delta n_{\mathbf{k}}(t) \delta n_{\mathbf{k}'}(0) \rangle = \vec{A}\,\vec{x}(t)$$

where

$$\delta n_{\mathbf{k}}(t) = n_{\mathbf{k}}(t) - \overline{n}_{\mathbf{k}}$$

and $A_{\mathbf{k}} = \frac{g}{\hbar V}(U_{\mathbf{k}} + V_{\mathbf{k}})^2$, $\quad x_{\mathbf{k}}(t) = \sum_{\mathbf{k}'} A_{\mathbf{k}'} \langle \delta n_{\mathbf{k}}(t) \delta n_{\mathbf{k}'}(0) \rangle$

To calculate $\vec{x}(t)$ we need description of mode occupation numbers $n_{\mathbf{k}}(t)$!

Kinetic equation

$$\dot{n}_{\mathbf{k}} = -\Gamma_{\mathbf{k}} n_{\mathbf{k}} + I(n_{\mathbf{k}})$$

where damping $\Gamma_{\mathbf{k}}$ and gain $I(n_{\mathbf{k}})$ rates are calculated within Bogoliubov method by using Fermi golden rule.

Important aspects:

- condensate treated as a isolated system
- Landau and Beliaev processes included: $k + q \longrightarrow q'$ and $k \longrightarrow q + q'$
- stationary solution: $\overline{n}_{\mathbf{k}} = \frac{k_B T}{\epsilon_{\mathbf{k}}}$

Further steps:

- linearize the kinetic equations to have equation for $\delta n_{\mathbf{k}}(t) = n_{\mathbf{k}}(t) - \overline{n}_{\mathbf{k}}, \quad \delta\dot{\vec{n}} = M\delta\vec{n}$
- get linearized equations for the correlation functions $\langle \delta n_{\mathbf{k}}(t) \delta n_{\mathbf{k}'}(0) \rangle$, and vector $\dot{\vec{x}} = M\vec{x}$

Analytical result

Asymptotic expression for the condensate phase variance:

$$\mathrm{Var}\,\varphi(t) \simeq \mathcal{A}t^2 + \mathcal{B}t + C \qquad \text{for} \qquad t \to \infty$$

with:

$$\mathcal{A} = \frac{1}{\hbar^2}\left(\frac{d\mu}{dE}\right)^2_{E=\bar{E}} \mathrm{Var}\,E$$

$$\mathcal{B} = -2\vec{A}\,M^{-1}\,\vec{X}(0) \qquad \text{and} \qquad C = -2\vec{A}\,M^{-2}\,\vec{X}(0)$$

- we found explicit expressions for $\mathcal{B}$ and C in terms of M and $t = 0$ correlation functions
- $\mathcal{B}$ and C do not depend on energy fluctuations in the initial state
- $\mathcal{A}$, $\mathcal{B}$ and C depend on temperature

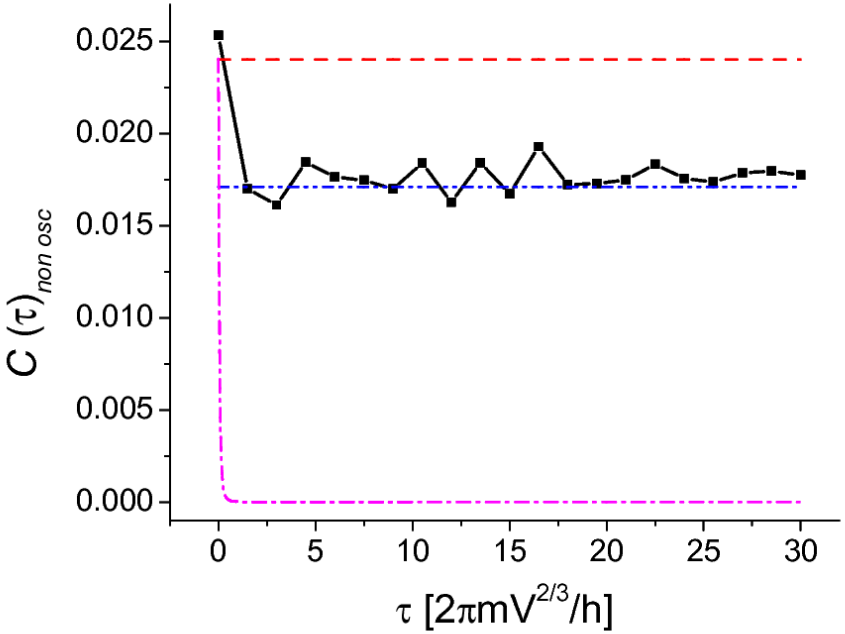
A. Sinatra, Y. Castin, E. Witkowska, Phys.Rev.A **80**, 033614 (2009)

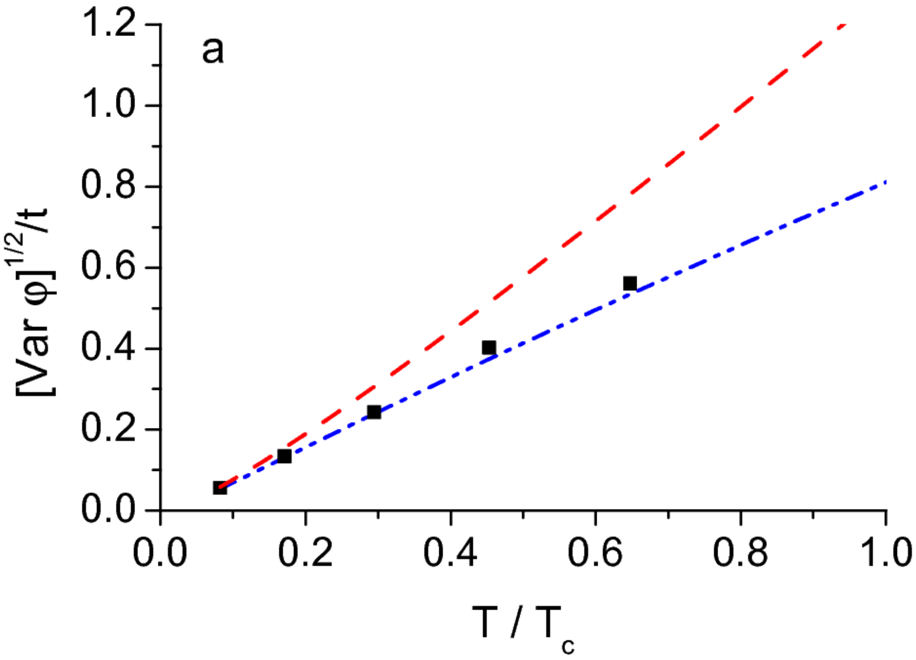
Ballistic spreading of the phase

Initial states $\psi(t=0)$ in canonical ensemble. Analytical treatment (blue lines) against classical fields simulations (points).

$C(t) \to \mathcal{A}_{\rm cl}$

$$\frac{\sqrt{\mathrm{Var}\,\varphi(t)}}{t} \simeq \sqrt{\mathcal{A}_{\rm cl}}$$

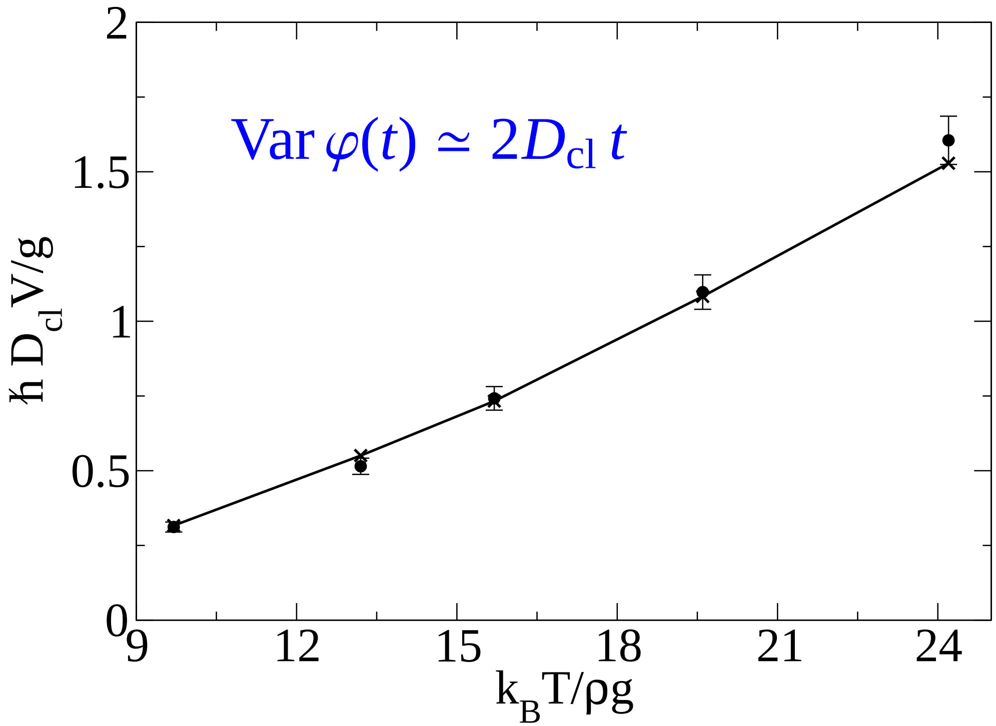




A. Sinatra, Y. Castin, E. Witkowska, Phys.Rev.A **75**, 033616 (2007)

Phase Diffusion

Initial states $\psi(t=0)$ in microcanonical ensemble. Analytical treatment ($\times$) against classical fields simulations ($\bullet$).



A. Sinatra, Y. Castin, Phys.Rev.A **78**, 053615 (2008)
A. Sinatra, Y. Castin, E. Witkowska, Phys.Rev.A **80**, 033614 (2009)

Quantum theory

Calculations within quantum theory can be found in:

A. Sinatra, Y. Castin, E. Witkowska, Phys.Rev.A 80, 033614 (2009).

Conclusions

- using **kinetic equations** we calculated how the variance of the phase grows in time for a nonzero temperature:

$$\mathrm{Var}\,\varphi(t) \simeq \mathcal{A}t^2 + \mathcal{B}t + C \qquad \text{for} \qquad t \to \infty$$

- **ballistic spreading** of the phase is present within canonical distribution of the initial states: $\mathcal{A} \propto \mathrm{Var}E$
- after suppression of the energy fluctuations phase undergo **diffusion motion** with $D = \mathcal{B}/2$
- **classical fields simulations** as a guideline and test of predictions of the quantum theory