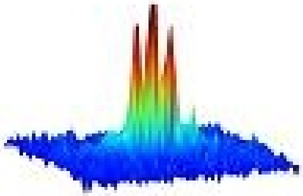
Many body dynamics of harmonically confined bright matter-wave solitons



Durham soliton theory group

Simon Gardiner

Supervisor

David Holdaway
(me)
PhD student

Tom Billam: Phd student

Christoph Weiss: Post Doc

John Helm: Phd student

Outline of talk

- Motivation and goals
- Reduced dimensional Bose gases: The difference between repulsive and attractive
- The limitations of the Gross Pitaevskii equation and the Bethe Ansatz
- Interacting Bosons in a harmonic oscillator potential
- Creating non local superpositions in a trap

Project Aims

Some questions to address:

- Does coherence persist between solitons in multisoliton systems (e.g. those forming from a collapsing Bose gas)? Or does the state fragment.
- How do solitons behave while interacting with one another? Quantitative effects of relative phase and external potential.
- How many atom's do we need for the classical field picture to work?
- Could atomic solitons be used to probe potentials via interferometry.

Like all good theorists we start with a Hamiltonian!

$$\hat{H} = \int d\mathbf{r}\,\hat{\Psi}^\dagger(\mathbf{r})\hat{H}_0\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\iint d\mathbf{r}d\mathbf{r}'\,\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}')$$

For simplicity, we assume a contact potential which reproduces the s-wave scattering length, this is found to give good agreement with experiment.

$$V(\mathbf{r}-\mathbf{r}') = V(\mathbf{R}) = \underbrace{\frac{4\pi\hbar^2 a_s}{m}}_{g_{3d}}\delta(\mathbf{R})\frac{\partial}{\partial\mathbf{R}}\mathbf{R}$$

In reduced dimension

- Reduced dimensionality arises when dynamics in one or more dimension are entirely determined by an external potential, with all the atoms occupying the same eigenstate of this potential.

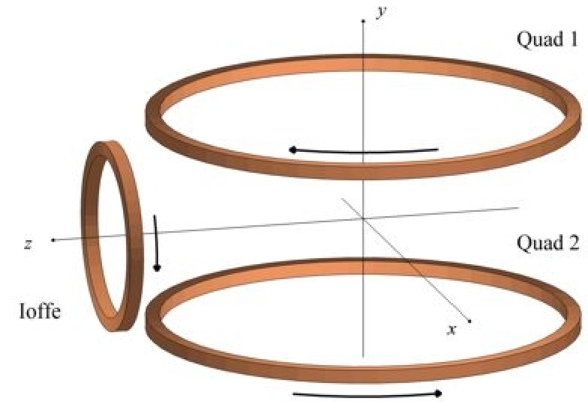
$$\text{e.g. } \Psi = \psi(x_1, .., x_n) \prod_{k=1}^{N} \phi(y_k, z_k)$$

- This dependence can simply be integrated out to give an effective 1D theory with an effective 1D coupling constant

$$g_{1d} = g_{3d} \iint dy\, dz\, |\phi(y,z)|^4$$

- For most experiments, this potential will be harmonic, if we consider a radially symmetric potential this reduces to.

$$g_{1d} = \frac{g_{3d}}{2\pi a_\perp} = \underline{2\hbar\omega_r a_s}$$



An Ioffe-Pritchard magnetic trap can create a radially symmetric harmonic potential for atoms

What do we need for this to be true?

- 1) Thermal fluctuations must be much smaller than the energy gaps between radial excited states:

$$k_b T \ll \hbar\omega_\perp$$

- 2) The chemical potential due to the interactions (which do not scale linearly with number) must also be much less than the radial spacing.

$$|\mu_{\text{int}}| \ll \hbar\omega_\perp$$

What it means to be in 1d

For repulsive gases $g_{1d} > 0$

- One can consider making an arbitrarily tight axial potential using a high powered, far detuned laser or magnetic trap.
- If the axial potential is also harmonic and the mean field strength is strong, the axial wavefunction can be well described by a Thomas Fermi profile.

$$|\mu_{\text{int}}| \propto (N g_{1d} \omega_x)^{2/3} m^{1/3} \ll \hbar\omega_\perp$$

$$\text{Therefore we require:} \quad N^2 a_s^2 m \omega_x^2 \ll \hbar\omega_\perp$$

What it means to be in 1d

For attractive gases $g_{1d} < 0$

- Even without an axial trapping potential the GPE predicts the ground state will be localised.
- If only a weak potential exists axially, the BEC will be well described by a classical soliton.

$$\phi(x) = \mathrm{sech}(x/2\xi)/2\xi^{1/2} \qquad \xi = \hbar^2/mg_{1d}(N-1)$$

$$N \gg 1 \rightarrow \ |\mu_{\mathrm{int}}| \propto \frac{mN^2 g_{1d}^2}{\hbar^2} \propto N^2 a_s^2 m\omega_\perp^2 \ll \hbar\omega_\perp$$

$$\text{Therefore we require: } \frac{N a_s}{\sqrt{\hbar/m\omega_\perp}} \ll 1$$

Consequences of this condition

$$\frac{N|a_s|}{a_\perp} \ll 1$$

(collapses if $\gtrsim 2/3$)

Given a particular N and a_s we CANNOT arbetrarily tighten the radial trapping to achieve a 1d system, quite the opposite!

To get into 1d we must therefore:

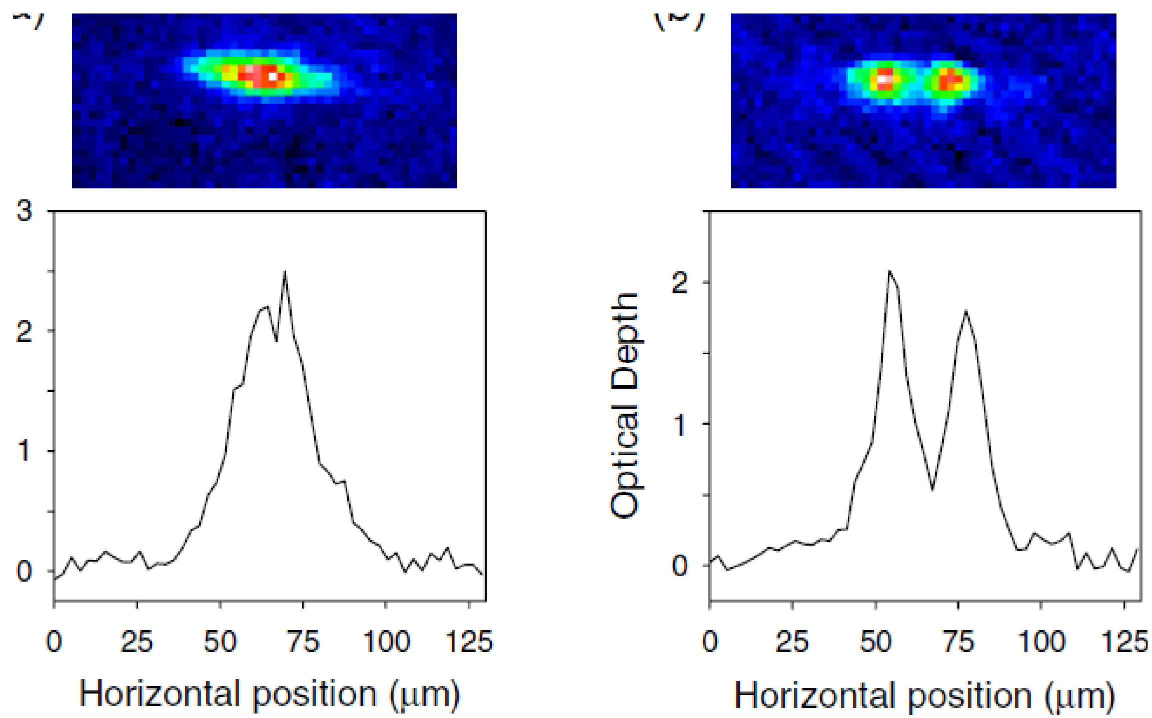
- 1) Use a limited number of atoms
- Problems: Poor statistics, losses will be significant
- 2) Tune the interactions to be very weak
- Problems: Require precise tuning, vulnerable to stray fields
- 3) Get the temperature very low, so the radial trapping can be weak: Also reduces 2 and 3 body loss effects!

Excited state stability

- Excited states can be stable against collapse even above this collapse threshold, when ground state would collapse.

- 3d GPE simulations predict the system will only be stable if the solitons form with a phase difference of around pi.

•Relative phase effects only the point of collision. The reason it is important is that regions of high density occur if the interference is constructive causing transfer to relative degrees of freedom.



[1] Cornish *et al*: *Phys. Rev. Lett.* **96**, 170401

Soliton units

- For the rest of the talk I will consider units in which the length scale is set my the soliton length.

$$\tilde{x} = x/\xi = x \times m g_{1d}(N-1)/\hbar^2$$

$$\tilde{t} = t \times m g_{1d}^2 (N-1)^2/\hbar^3$$

$$\hbar = m = (N-1) g_{1d} = 1$$

- I will also assume this condition for separation of the wave function, required to be in a 1d regime, is satisfied.

Bethe ansatz in 1d

- The many body system in free space is exactly solvable. The eigenstates are multiple cluster type states, with a free (real) momentum parameter for each cluster and a phase jump at locations where the clusters meet.

$$\psi = \mathcal{N} \sum_{\mathcal{P}} \prod_{j=1}^{N} \prod_{k=1}^{j-1} \left(\frac{1}{2} + \frac{\mathrm{i}\ \mathrm{sign}(x_k - x_j)}{2(N-1)(p_k - p_j)} \right) \exp\left(\sum_k \mathrm{i} p_k x_k \right)$$

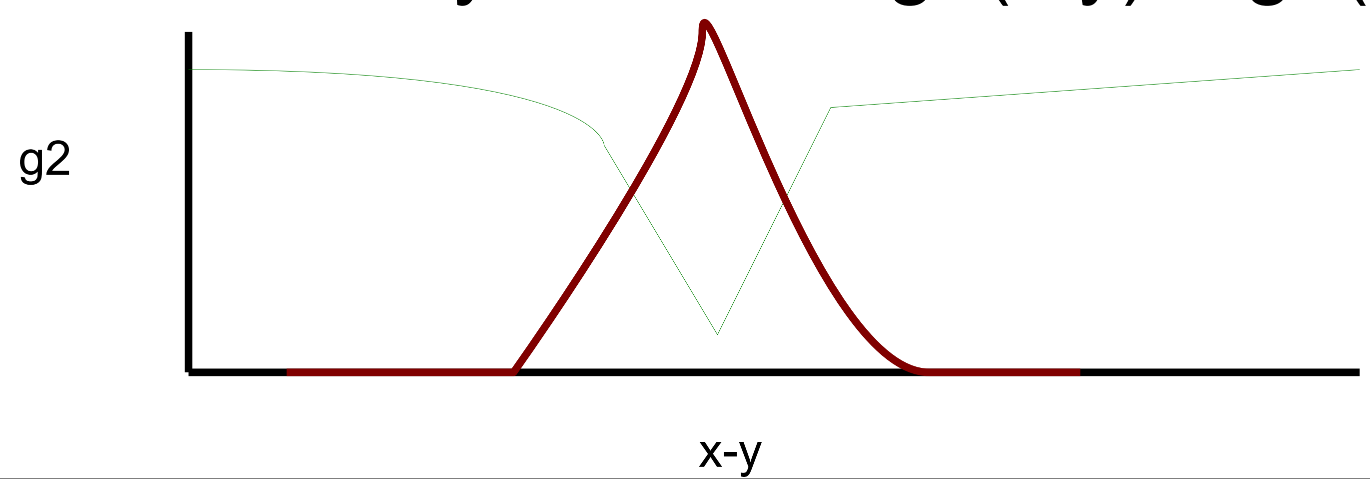
$$p_k = \begin{cases} P_1 + \frac{\mathrm{i}g}{2}(n_1 + 1 - 2k) & 1 \leq k \leq n_1 \\ P_2 + \frac{\mathrm{i}g}{2}(n_1 + n_2 + 1 - 2k) & n_1 \leq k \leq n_2 \\ \ldots\ldots \end{cases}$$

- One can consider taking superpositions of eigenstates with different P_ℓ to create (initially) localised multisoliton states.
- Ground state of the form: exp(-sum_(k<j) | x_k – x_j | / 2(N-1))

[2] Lai, Y. and Haus, H. A., *Phys. Rev. A*, **40**, 854 (1989)

Ground state correlations

- Consider atoms in a ring trap, Bethe ansatz still produces solutions.
- The density of ground states will always be uniform about the ring but correlated. The GPE assumes the correlations are unity.
- Translationally invariant: g2(x,y) = g2(|x-y|)



Green: Finite Repulsive interactions
Red:Finite attractive interactions

Dynamics of the centre of mass

AtMol
Durham Atomic and Molecular Physics

- The interactions do not affect the centre of mass of the Bose gas. Hence we have:

$$\mathrm{H} = \mathrm{H}_{\mathrm{cm}} + \mathrm{H}_{\mathrm{rel}}, \qquad [\mathrm{H}_{\mathrm{cm}}, \mathrm{H}_{\mathrm{rel}}] = 0$$

$$\mathrm{H}_{\mathrm{cm}} = -\frac{1}{2N}\frac{\partial^2}{\partial x_{\mathrm{cm}}^2} \qquad x_{\mathrm{cm}} = \frac{1}{N}\sum_{k=1}^{N} x_k$$

- As the Hamiltonian can be split into two commuting parts, the evolution is separate and the eigenstates of each one can simply be multiplied together.

- The centre of mass behaves like a free particle of mass N m

Comparing GPE and many body

- Despite the fact the evolution of these two states will be different, one would expect initially these wave functions would agree for high N.
- Wavepackets which are initially Gaussian profiles stay in a Gaussian profile and so are a natural choice.
- If we were to take a momentum superposition of the form:

$$\psi(x,t) = \mathcal{N} \int \exp(-\alpha^2 p^2/2) \psi_{\text{free}}(x_1, .., x_N, p) \exp(i(p^2 t/2 + E_0 t))$$

- What number and parameter alpha would be required for this to approximate a mean field state?

Comparing GPE and many body

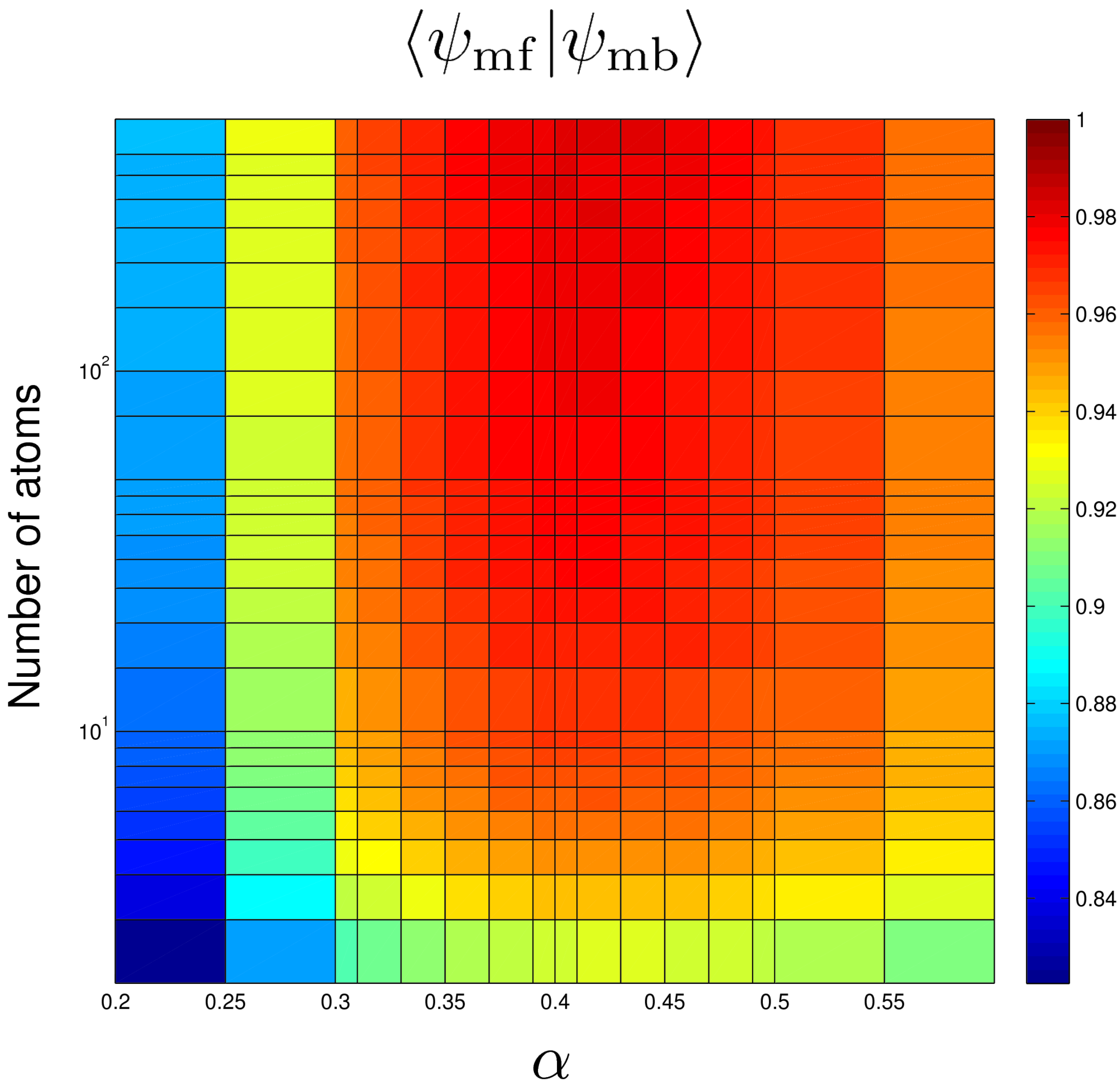
- The GPE ground state is a product form of sech functions.
- Such states are generally more resilient to atom loss.
- If we were to take a momentum superposition of the form:

$$\psi(x,t) = \mathcal{N} \int \exp(-\alpha^2 p^2/2) \psi_{\text{free}}(x_1, .., x_N, p) \exp(i(p^2 t/2 + E_0 t))$$

i.e. the centre of mass described by a Gaussian wavepacket, the state is localised in space, but the centre of mass will spread in time. The importance of this form will be apparent later.

- It is interesting to see when this initial state (t=0) agrees with the GPE prediction, hence we compare the two with a fidelity measurement.

Comparing GPE and many body



Overlap between the mean field state and the initial many body ansatz. Best overlap is achieved on the line alpha = 0.4.

This result indicates the centre of mass expectation value of the classical soliton looks like a single atom wavefunction.

Evolution of the centre of mass

$$f_{\rm cm}(x_{\rm cm},t) = \int dp\, \tilde{N} \exp\left(ipx_{\rm cm}\sqrt{N} - \frac{ip^2 t}{2} - \frac{p^2}{2\alpha^2}\right)$$
$$= \left(\frac{\alpha}{\sqrt{\pi(1+t^2\alpha^4)}}\right)^{1/2} \exp\left(-\frac{N\alpha^2 x_{\rm cm}^2}{2(1+it\alpha^2)}\right)$$

- Centre of mass wavefunction spreads with time.
- Typical “soliton times” are 10-100 ms
- Time to double in width $t = \alpha^{-2}$

$$\sigma^2 = \left(\alpha^{-2} + t^2\alpha^2\right)/N$$

An experimental scenario

AtMol
Durham Atomic and Molecular Physics

- The most likely scheme to create solitons would be to evaporatively cool the atoms in a dipole trap with repulsive interactions to the ground state and then adiabatically tune the interactions to attractive.

$$H_{\mathrm{trap}} = \alpha^4 x^2/2 \qquad \alpha = \xi/a_x = \sqrt{\hbar^3 \omega_x/m}/g_{1d}(N-1)$$

- In a harmonic potential, the centre of mass can be separated as before, giving rise to the so called Kohn mode.

- Such a procedure would not effect the centre of mass, which would lie in the ground state of the trap.

$$\psi_{\mathrm{trap}} = \psi_{\mathrm{rel}} \times \psi_{\mathrm{cm}}(x_{\mathrm{cm}}) \qquad \psi_{\mathrm{cm}} = \mathcal{N} \exp\left(-\frac{N\alpha^2 x_{\mathrm{cm}}^2}{2}\right)$$

Energy Level shifts

- We can consider the effect of the potential on the relative degrees of freedom on the ground state as a weak perturbation

$$E^{(1)} = \frac{\alpha^4}{2}\langle\psi|\int dx\ \hat{\Psi}^\dagger(x)\hat{\Psi}(x)x^2|\psi\rangle = \frac{\alpha^2}{2} + \frac{\alpha^4(N-1)^2}{N}\sum_{k=1}^{N-1}\frac{1}{k^2}$$

$$\text{recall } \alpha = \frac{\text{Soliton length}}{\text{harmonic length}}$$

$$N >> 1$$

$$\approx \alpha^4\left[(N-2)\frac{\pi^2}{6} - 1\right] + \frac{\alpha^2}{2}$$

- The energy of the many body solutions + this correction is always smaller than the corrections to the mean field solution.

- When the difference between the exact energy of the ground state and this correct is much less than the chemical potential, we are in the “soliton like regime” where free results are likely applicable.

Energy Level shifts

- We can estimate an lower bound on how much of the relative component of the wavefunction is in the free ground state from energy considerations.
- We take an ansatz for the relative state in the trap that will maximise depletion for a given energy:

$$|\psi\rangle_{\mathrm{rel}} = a|0\rangle_{\mathrm{rel}} + \sqrt{1-a^2}|1\rangle_{\mathrm{rel}}$$

- Giving us an upper bound determined by

$$\langle\psi|\hat{H}_{\mathrm{total}} - \hat{H}_{pot}|\psi\rangle_{\mathrm{rel}} = a^2 E_0 + (1-a^2)(E_0 + \mu_{\mathrm{free}})$$

$$a^2 \geq 1 - (E_{\mathrm{in\ trap}} - E_{\mathrm{pot}} - E_0)/\mu_{\mathrm{free}}$$

The numerical procedure in brief

AtMol
Durham Atomic and Molecular Physics

- To calculate wavefunctions and energies in the many body system I expand the field operators over a basis set of Hermite functions.

$$\hat{\Psi}(x) = \sum_k \hat{a}_k \psi_k(wx)$$

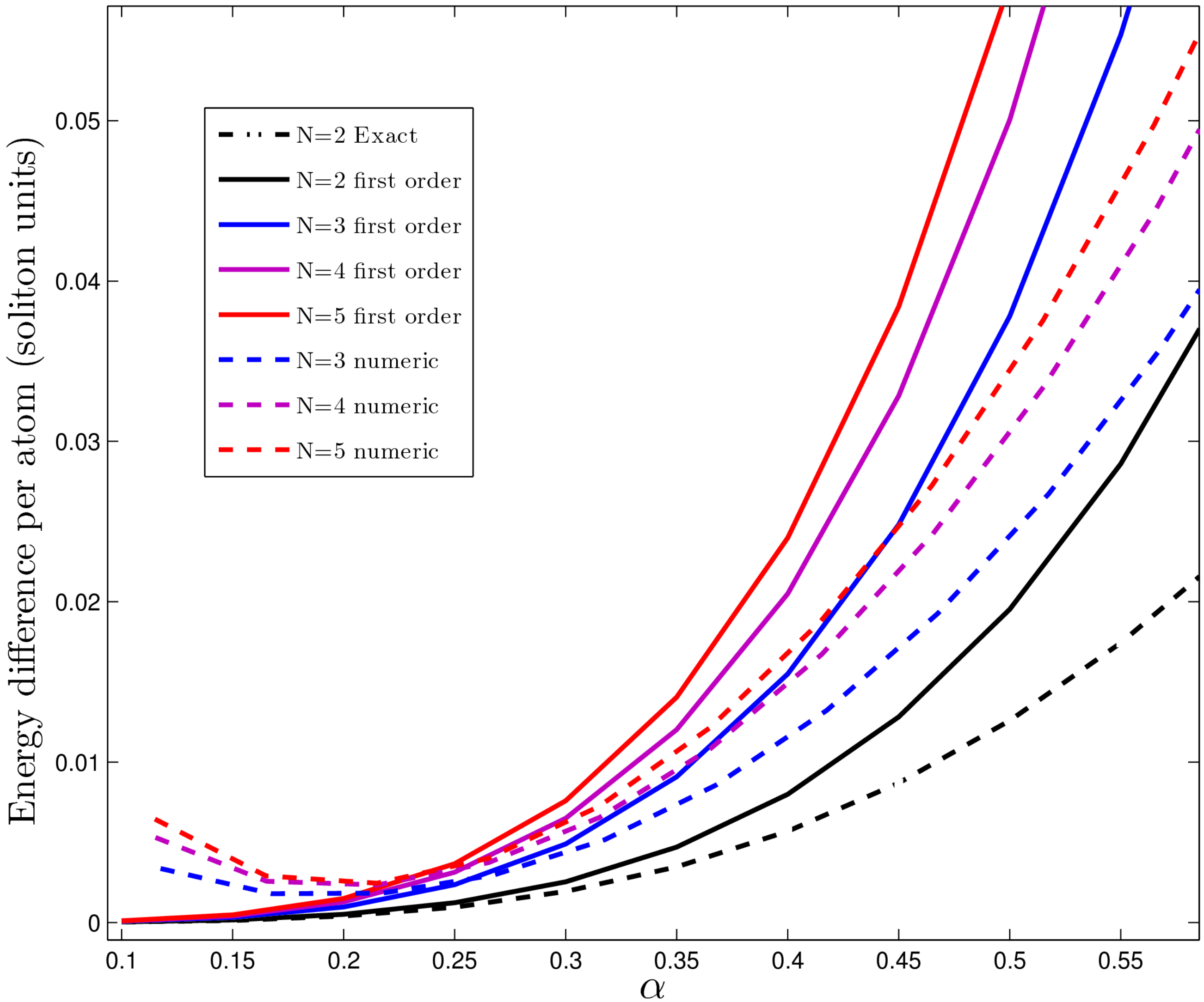
- This gives a basis set of number occupation kets, truncated to some cut off energy K.

$$\{|N,0,0,...\rangle, |N-1,1,0,...\rangle, ...\}$$

- To reduce the basis size I diagonalise the basis in terms of the number operator for centre of mass excitation and keep only states with eigenvalue zero. $M = \langle a|\hat{\mathcal{L}}^\dagger_{\mathrm{cm}}\hat{\mathcal{L}}_{\mathrm{cm}}|b\rangle$

- Then I diagonalise the truncated Hamiltonian

Ground state energies in the trap

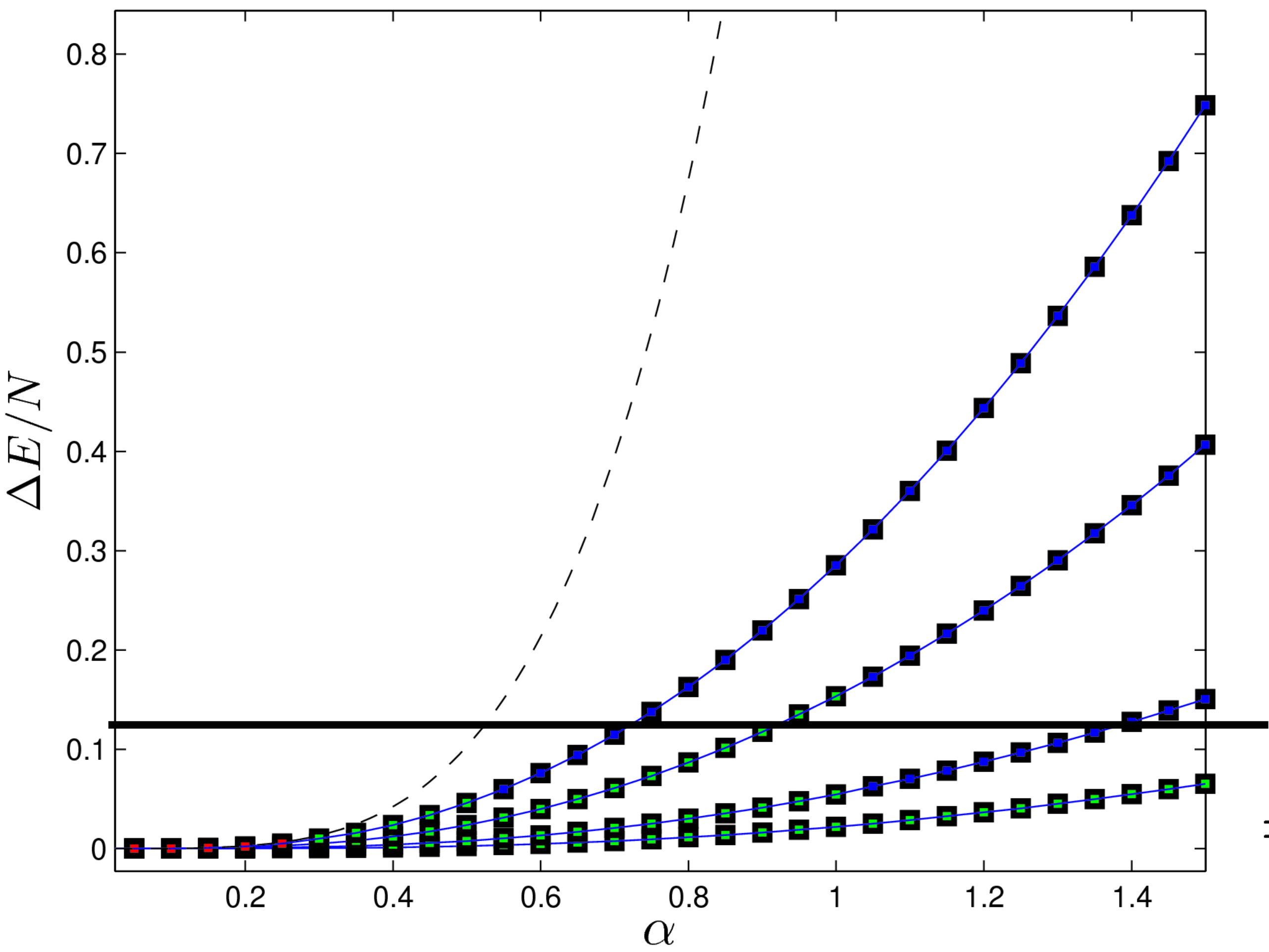


Energy of the relative wavefunction groundstate in a Harmonic trap.

My numerical method breaks down too early using bare harmonic oscillator states as a basis set. Energies increase as potential drops which is highly non-physical

Ground state energies in the trap

AtMol
Durham Atomic and Molecular Physics



$$\Delta E = E_{\text{in trap}} - E_0$$

Energy per atom for the ground state for N =2,3,6,10 respectively. Dotted line is perturbation correction to mean field.

Colours of square indicate the method used,

Red: Perturbation theory

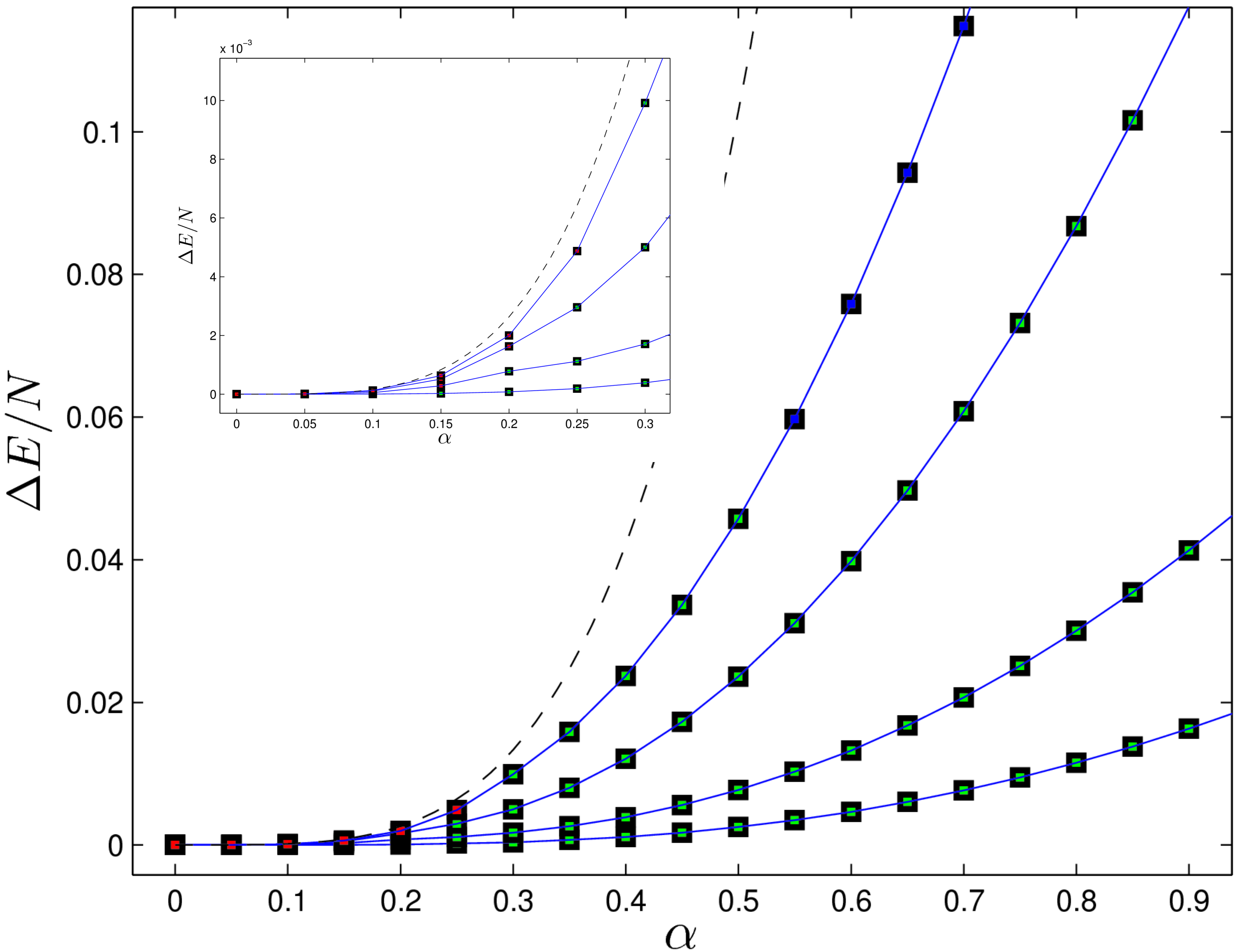
Green: Fixed width basis

Blue: Bare eigenstates

$$|\mu_{\text{min}}| = \frac{1}{8}$$

$$= \lim_{N \to \infty}(\mu)$$

Ground state energies in the trap



$$\mu = -\frac{N}{8(N-1)}$$

Dependence is initially quartic w.r.t. alpha but goes as alpha^2 asymptotically

Scattering off a delta potential

- The centre of mass separation gives new possibilities to create superposition states.
- We can consider a quantum soliton undergoing SHM in an axial trapping potential.

$$\psi(\mathbf{x},t) = \psi_{\text{rel}} \times \exp\left(-\frac{|\beta|^2}{2}\right)\sum_{k=0}^{\infty}\frac{\beta^k}{\sqrt{k!}}\phi_k(x_{\text{cm}})$$

- At a time t=0 a delta function like potential is switched on in the trap centre.

Scattering off a delta potential

- Fast scattering regime:
  - If the soliton is moving sufficiently fast we can consider the evolution of the wavefunction on collision to be approximately linear.
  - This process would be well described by the GPE as the potential strongly couples the centre of mass wavefunction to the internal degrees of freedom.
  - Some theoretical work is being done within the Durham group on using this effect to split and recombine solitons.
  - It is found that the amount either side after recombination depends on the relative phase and velocity in a predictable way, hence they can be used as a probe of (slowly varying) potentials.

Scattering of a delta potential

- Slow scattering regime:
  - If the soliton is moving sufficiently slowly, classical field pictures predict there can be no reflection from this barrier.
  - This condition requires that the kinetic energy at the trap centre should be smaller than the chemical potential

$$\frac{|\beta|^2}{2} < \frac{|\mu|}{\alpha^2}$$

  - However the centre of mass component can still scatter through the barrier.

Scattering of a delta potential

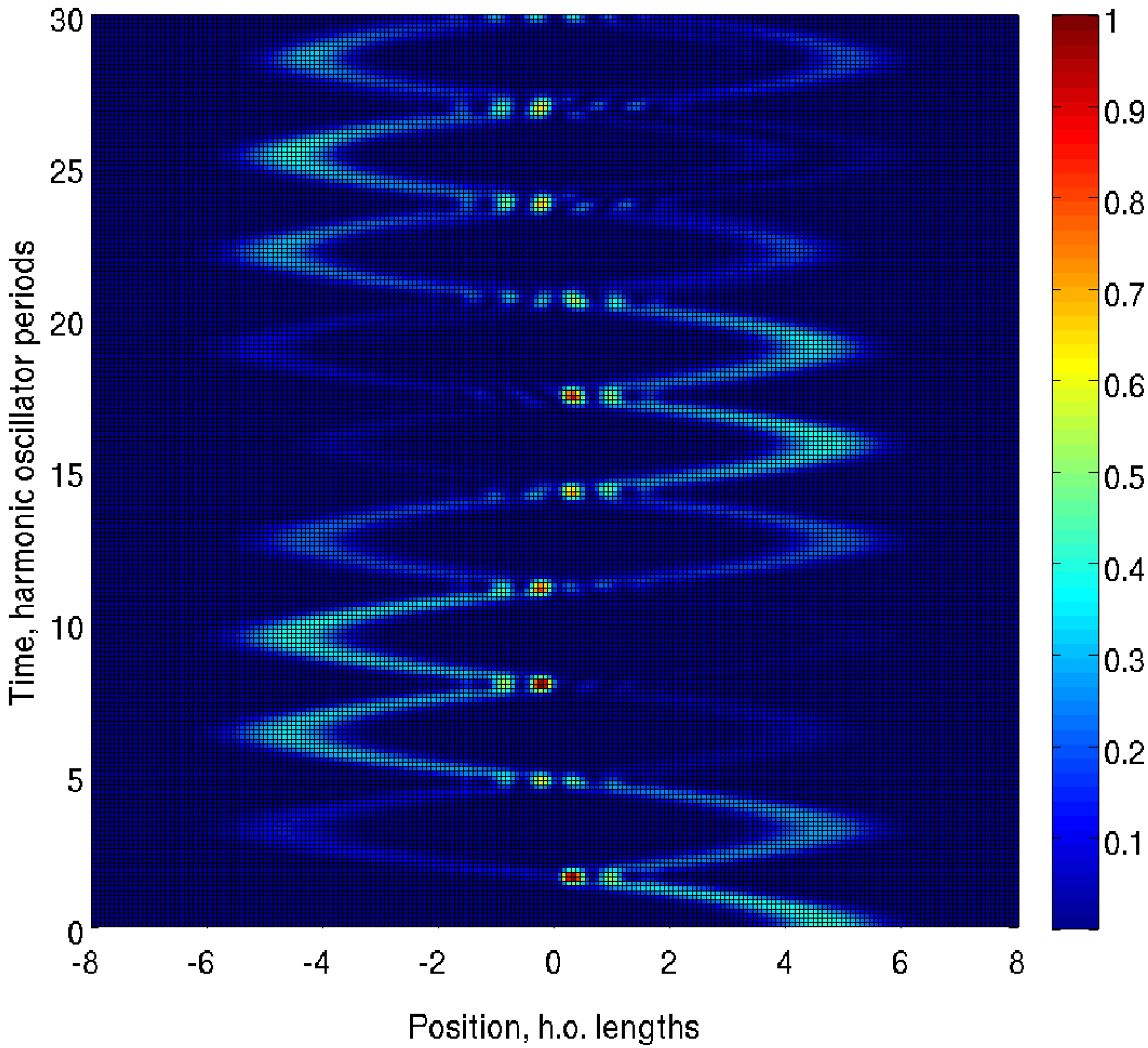
- The centre of mass wavefunction will in fact scatter strongly off this defect.
- The relative degrees of freedom will only be weakly effected and weakly coupled to the centre of mass. Hence we approximate

$$\sum_{k=1}^{N} \delta(x_k) \approx \delta\left(\sum_{k=1}^{N} x_k\right)$$

- The new centre of mass eigenstates of this system are known, they are Tricomi hypergeometric functions.

$$\varphi_k(x) = \mathcal{N}_k \, \mathrm{U}(-\mathrm{v_k}, 1/2, \mathrm{N}\mathrm{x}_{\mathrm{cm}}^2 \alpha^2) \exp\left(-\mathrm{N}\mathrm{x}_{\mathrm{cm}}^2 \alpha^2/2\right)$$

Scattering of a delta potential



Centre of mass wavefunction density (arbitrary units). Quantities are scaled to harmonic units

$$\beta = 3$$

$$V_0 = 10$$

Scattering of a delta potential

AtMol
Durham Atomic and Molecular Physics

- These mesoscopic superposition states could in theory be observed.
- Imaging at a specific time would be expected to show a soliton at a random position with the probability determined by the centre of mass wavefunction.
- It would also be possible to release the trap and observe interference between the wavefunction at each side such as was suggested in [5]

[5] Weiss, C and Castin, Y, *Phys. Rev. Lett.*, **102**, 010403 (2009)

Summary

- Attractive interactions allow the possibility of self trapped states and are only stable below a certain collapse threshold, which is also relevant with the condition to being 1dimensional objects.
- The GPE is very good at describing the internal dynamics of solitons for a normal number of atoms.
- Interesting quantum effects arise from the fact that the interactions do not affect the behaviour of the centre of mass.
- The internal structure of quantum solitons is unaffected by small harmonic external potentials.
- Good understanding can be achieved by treating quantum solitons as individual particles where the coupling between internal degrees of freedom and the centre of mass is weak.

With thanks to

Outlook

- Use the many body code to simulate time dependant dyanmics e.g.
  - Soliton formation in trap via feshbach tuning
  - Delta function scattering in the intermediate regime
- Impact of inelastic scattering and observation, robustness of delocalised state.
- Investigate soliton wavepacket splitting for finite width potentials.

References

[1] Cornish *et al*: *Phys. Rev. Lett.* **96**, 170401

[2] Lai, Y. and Haus, H. A., *Phys. Rev. A*, **40**, 854 (1989)

[3] Haugset, Tor *etal Phys. Rev. A*, **57**, 3809 (1998)

[4] Astrakharchik, G. E. and Giorgini, S. *Phys. Rev. A*, **68**, 031602

[5] Weiss, C and Castin, Y, *Phys. Rev. Lett.*, **102**, 010403 (2009)

Image: http://www.physics.otago.ac.nz/uca/resources/collisions/node5.html

With thanks to

EPSRC

Engineering and Physical Sciences Research Council