Disordered bosons:
condensate and excitations

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[PRA **83**, 063629 (2011); arXiv:1101.4781]

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The disordered Bogoliubov problem

$$\hat{H} = \int \mathrm{d}^d r \, \hat{\Psi}^\dagger \left[\frac{-\hbar^2}{2m}\nabla^2 - \mu + \frac{g}{2}\hat{\Psi}^\dagger\hat{\Psi} + V(\mathbf{r}) \right] \hat{\Psi}$$

SF phase: Bose statistics > interaction > disorder

- Bogoliubov (1947): $\hat{\Psi} = \Phi + \delta\hat{\Psi}$
- Where is the problem? $V(\mathbf{r})$ random: laser speckle

arXiv:0804.1621 [pdf]

Direct observation of Anderson localization of matter-waves in a controlled disorder
Juliette Billy (LCFIO), Vincent Josse (LCFIO), Zhanchun Zuo (LCFIO), Alain Bernard (LCFIO), Ben Hambrecht (LCFIO), Pierre Lugan (LCFIO), David Clément (LCFIO), Laurent Sanchez-Palencia (LCFIO), Philippe Bouyer (LCFIO), Alain Aspect (LCFIO)

arXiv:1105.5368 [pdf, other]

Three-Dimensional Anderson Localization of Ultracold Fermionic Matter
S. S. Kondov, W. R. McGehee, J. J. Zirbel, B. DeMarco

arXiv:1108.0137 [pdf, ps, other]

Three-dimensional localization of ultracold atoms in an optical disordered potential
Fred Jendrzejewski (LCFIO), Alain Bernard (LCFIO), Killian Mueller (LCFIO), Patrick Cheinet (LCFIO), Vincent Josse (LCFIO), Marie Piraud (LCFIO), Luca Pezzé (LCFIO), Laurent Sanchez-Palencia (LCFIO), Alain Aspect (LCFIO), Philippe Bouyer (LCFIO, LP2N)


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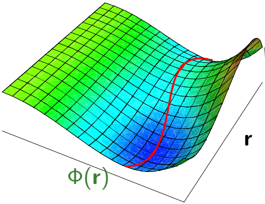
Bräu
Rupert Ebner Tierarzt
Cafe Bistro West
A
Specklestraße

Inhomogeneous Bogoliubov theory

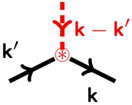
$$\hat{\Psi}(\mathbf{r}) = \Phi(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r})$$



1. Expand $\hat{H}$ around deformed ground state $\Phi(\mathbf{r}) = \Phi[V(\mathbf{r})]$
2. Bring excitation Hamiltonian into impurity-scattering form

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_k \hat{\Gamma}^\dagger_{\mathbf{k}} \hat{\Gamma}_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}'} \hat{\Gamma}^\dagger_{\mathbf{k}} \mathcal{V}_{\mathbf{k}\mathbf{k}'} \hat{\Gamma}_{\mathbf{k}'}$$

$$\overset{g=0}{\mapsto} \sum_{\mathbf{k}} \epsilon^0_k \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'}$$



3. Compute physical quantities using machinery of perturbation theory. Potential correlation

$$\circledast \underset{\mathbf{q}}{\cdots} \circledast = \overline{|V_{\mathbf{q}}|^2} = V^2 \sigma^d C_d(q\sigma), \qquad \sigma \approx 1\,\mu\text{m}$$

- Examples:
  condensate depletion (figure of merit)
  quasiparticle dispersion (sound velocity, DOS, ...)

1. Saddlepoint expansion

Ground state: $\Phi(\mathbf{r}) = \sqrt{n(\mathbf{r})}$ solves Gross-Pitaevskii eq.

$$[-\frac{\hbar^2\nabla^2}{2m} - \mu + g|\Phi(\mathbf{r})|^2]\Phi(\mathbf{r}) = -V(\mathbf{r})\Phi(\mathbf{r})$$

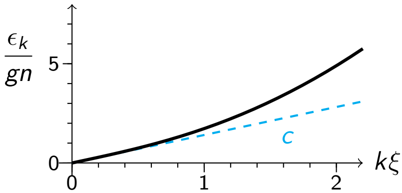
Excitations:

$$\begin{aligned}\hat{\Psi}(\mathbf{r}) &= e^{i\delta\hat{\varphi}(\mathbf{r})}\sqrt{n(\mathbf{r}) + \delta\hat{n}(\mathbf{r})} \\ &\approx \Phi(\mathbf{r}) + \frac{1}{2\Phi(\mathbf{r})}\delta\hat{n}(\mathbf{r}) + i\phi(\mathbf{r})\delta\hat{\varphi}(\mathbf{r})\end{aligned}$$

Homogeneous case $V = 0$:

$$\hat{\gamma}_{\mathbf{k}} = \frac{1}{2a_k\sqrt{n}}\,\delta\hat{n}_{\mathbf{k}} + ia_k\sqrt{n}\,\delta\hat{\varphi}_{\mathbf{k}}$$

$$a_k^2 = \epsilon_k^0/\epsilon_k, \quad \epsilon_k = \sqrt{\epsilon_k^0(\epsilon_k^0 + 2gn)}$$



$$\hat{H} = \sum_{\mathbf{k}} \epsilon_k \hat{\gamma}_{\mathbf{k}}^\dagger \hat{\gamma}_{\mathbf{k}}, \qquad [\hat{\gamma}_{\mathbf{k}}, \hat{\gamma}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$$

- Inhomogeneous BEC $\Phi[V(\mathbf{r})]$:

$$\hat{H} = \int \mathrm{d}^d r \left\{ \frac{\hbar^2}{2m} \left[\left(\boldsymbol{\nabla} \frac{\delta \hat{n}}{2\Phi} \right)^2 + \frac{(\nabla^2 \Phi)}{4\Phi^3} \delta \hat{n}^2 + \Phi^2 (\boldsymbol{\nabla} \delta \hat{\varphi})^2 \right] + \frac{g}{2} \delta \hat{n}^2 \right\}$$

- Scattering Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_k \hat{\gamma}^\dagger_{\mathbf{k}} \hat{\gamma}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} \left(\hat{\gamma}^\dagger_{\mathbf{k}}, \hat{\gamma}_{-\mathbf{k}} \right) \begin{pmatrix} W_{\mathbf{k}\mathbf{k}'} & Y_{\mathbf{k}\mathbf{k}'} \\ Y_{\mathbf{k}\mathbf{k}'} & W_{\mathbf{k}\mathbf{k}'} \end{pmatrix} \begin{pmatrix} \hat{\gamma}_{\mathbf{k}'} \\ \hat{\gamma}^\dagger_{-\mathbf{k}'} \end{pmatrix}$$

- Nambu spinors: $\hat{\Gamma}^\dagger_{\mathbf{k}} = \frac{1}{\sqrt{2}} \left(\hat{\gamma}^\dagger_{\mathbf{k}}, \hat{\gamma}_{-\mathbf{k}} \right)$

$$\boxed{\hat{H} = \sum_{\mathbf{k}} \epsilon_k \hat{\Gamma}^\dagger_{\mathbf{k}} \hat{\Gamma}_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}'} \hat{\Gamma}^\dagger_{\mathbf{k}} \mathcal{V}_{\mathbf{k}\mathbf{k}'} \hat{\Gamma}_{\mathbf{k}'}}$$

- Excitations

$$\delta\hat{\Psi}(\mathbf{r}) = \sum_{\mathbf{k}} [u_{\mathbf{k}}(\mathbf{r})\hat{\gamma}_{\mathbf{k}} - v^*_{\mathbf{k}}(\mathbf{r})\hat{\gamma}^\dagger_{\mathbf{k}}]$$

  expressed in a basis of deformed plane waves:

$$\begin{matrix} u_{\mathbf{k}}(\mathbf{r}) \\ v_{\mathbf{k}}(\mathbf{r}) \end{matrix} = \frac{1}{2}\left(\frac{\Phi(\mathbf{r})}{a_k\sqrt{n}} \begin{matrix} + \\ - \end{matrix} \frac{a_k\sqrt{n}}{\Phi(\mathbf{r})} \right) e^{i\mathbf{k}\cdot\mathbf{r}}$$

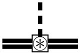
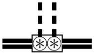
  Satisfies bi-orthogonality

$$\int \mathrm{d}^d r [u^*_{\mathbf{k}}(\mathbf{r}) u_{\mathbf{k}'}(\mathbf{r}) - v^*_{\mathbf{k}}(\mathbf{r}) v_{\mathbf{k}'}(\mathbf{r})] = \delta_{\mathbf{k}\mathbf{k}'},$$

  including the zero mode $\Phi[V(\mathbf{r})]$!
- $\mathcal{V} = \left(\begin{smallmatrix} W & Y \\ Y & W \end{smallmatrix}\right)$ exact in disorder!
- Smoothing [Sanchez-Palencia 2006]: expansion in $v = V/gn \ll 1$

$$\Phi[V(\mathbf{r})] = \sqrt{n} + \Phi^{(1)}(\mathbf{r}) + \Phi^{(2)}(\mathbf{r}) + \ldots$$

$$\mathcal{V} = \quad \mathcal{V}^{(1)} + \quad \mathcal{V}^{(2)} \quad + \ldots$$

=  +  + ...

2. Condensate depletion induced by disorder

- Non-condensed density $\delta n := L^{-d} \int \mathrm{d}^d r \left\langle \delta\hat{\Psi}(\mathbf{r})^\dagger \delta\hat{\Psi}(\mathbf{r}) \right\rangle$

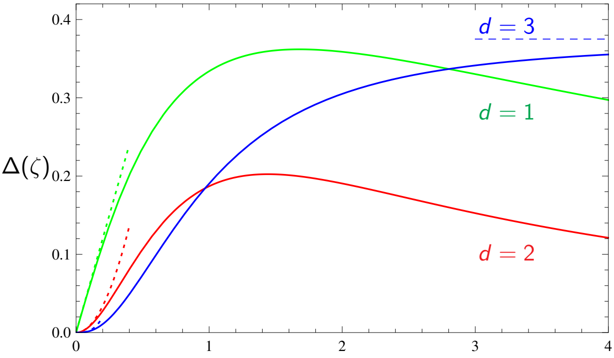
$$\delta n = \frac{1}{4nL^d} \sum_{\mathbf{k},\mathbf{k}'} \left\{ \left[a_k a_{k'} \check{n}_{\mathbf{k}'-\mathbf{k}} + \frac{n_{\mathbf{k}'-\mathbf{k}}}{a_k a_{k'}} \right] \left\langle \hat{\gamma}_{\mathbf{k}} \hat{\gamma}^\dagger_{\mathbf{k}'} + \hat{\gamma}^\dagger_{\mathbf{k}'} \hat{\gamma}_{\mathbf{k}} \right\rangle \right.$$
$$\left. + \left[a_k a_{k'} \check{n}_{\mathbf{k}'-\mathbf{k}} - \frac{n_{\mathbf{k}'-\mathbf{k}}}{a_k a_{k'}} \right] \left\langle \hat{\gamma}_{\mathbf{k}} \hat{\gamma}_{-\mathbf{k}'} + \hat{\gamma}^\dagger_{\mathbf{k}'} \hat{\gamma}^\dagger_{-\mathbf{k}} \right\rangle - 2\delta_{\mathbf{k}\mathbf{k}'} \right\}$$

- Get condensate profile $n(\mathbf{r})$ and $\check{n}(\mathbf{r}) = 1/n(\mathbf{r})$ from GPE
- Get expectation values $\left\langle \hat{\gamma}^\dagger_{\mathbf{k}'} \hat{\gamma}_{\mathbf{k}} \right\rangle$ etc. from Matsubara-Green
- Clean system at $T = 0$:

$$\delta n^{(0)} = \frac{1}{4} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[a_k - a_k^{-1} \right]^2 = \frac{c_d}{\xi^d} \ll n$$

Disorder-induced condensate depletion at $T = 0$:

$$\overline{\delta n}^{(2)} =: \delta n^{(0)} v^2 \Delta(\frac{\sigma}{\xi}) \ll n$$



white noise $\leftarrow$ $\zeta = \sigma/\xi$ $\rightarrow$ Thomas-Fermi

Primary effect of disorder: condensate deformation
Secondary effect: condensate depletion. Bogoliubov theory rules!

3. Disorder-averaged excitation spectrum

- Retarded Nambu-Green function

$$\mathcal{G}_{\mathbf{k}\mathbf{k}'}(t) := \frac{\Theta(t)}{i\hbar} \left\langle \left[\hat{\Gamma}_{\mathbf{k}}(t), \hat{\Gamma}^{\dagger}_{\mathbf{k}'}(0) \right] \right\rangle$$

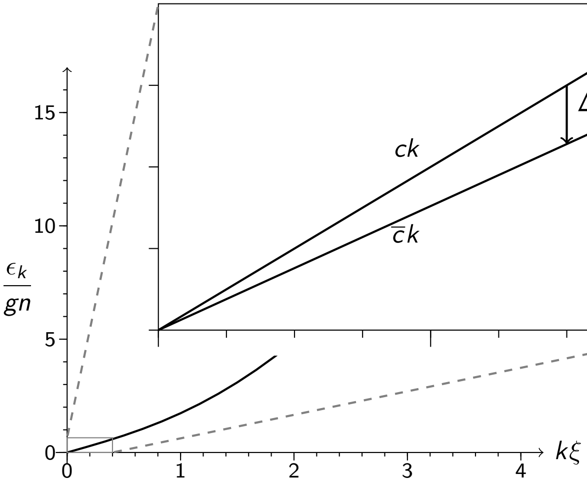
- Dyson: $\overline{\mathcal{G}} = \left[\mathcal{G}_0^{-1} - \Sigma \right]^{-1}$

Self-energy $\Sigma^{(2)} =$  $+$ 

- Renormalized dispersion

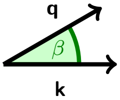
$$\bar{\epsilon}_k = \epsilon_k + \Sigma_{11}(k, \bar{\epsilon}_k)$$

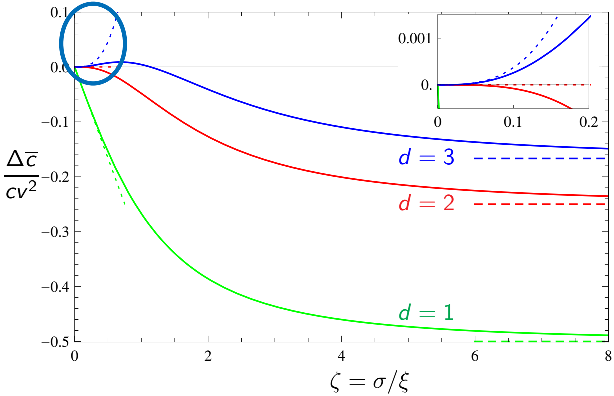
Renormalized speed of sound



Disorder-shift of sound velocity at $k\xi \to 0$:

$$\frac{\Delta \overline{c}}{c} = 2v^2 \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^2\xi^2 - (2+q^2\xi^2)\cos^2\beta}{(2+q^2\xi^2)^3} \sigma^d C_d(q\sigma),$$



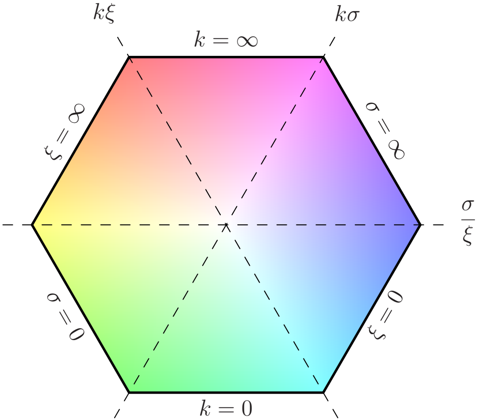


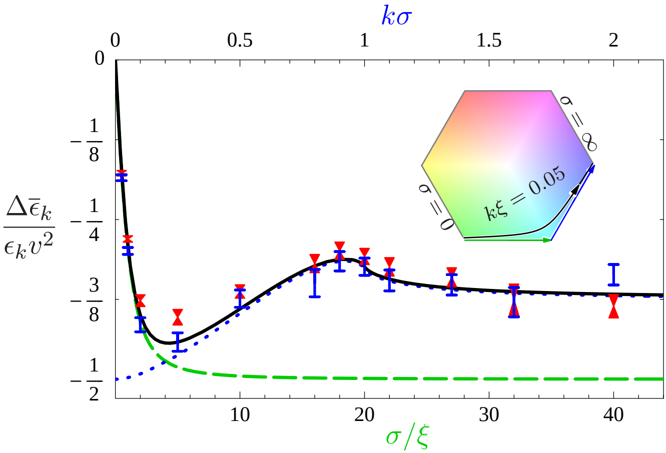
$$\frac{5C_3(0)\zeta^3}{48\sqrt{2}\pi}$$

- $\Delta \overline{c} < 0$
- Correlated disorder $\neq$ white noise [Giorgini, Pitaevskii, Stringari '94].

Parameter space

- Condensate healing length $\xi = \hbar/\sqrt{2gn}$
- Speckle correlation length σ
- Excitation wavelength $\lambda = 2\pi/k$





- Data points: t-dependent GP in red/blue detuned Speckle
- LDA $c(\mathbf{r}) = \sqrt{g/m}\,\Phi(\mathbf{r})$ cannot
- But weak lattices also can [Taylor & Zaremba 2003, Liang at al. 2008]

Conclusions

- $\hat{H} = \epsilon \hat{\Gamma}^\dagger \hat{\Gamma} + \hat{\Gamma}^\dagger \mathcal{V} \hat{\Gamma}$
- true quantum depletion $\langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi} \rangle_{\mathrm{disorder}}$
- $\Sigma^{(2)} =$  $+$ 
  - [Gaul & Müller, arXiv:1009.5448 and arXiv:1101.4781 (PRA)]
- To do:
  superfluid fraction,
  $T > 0$,
  ...