Collectibility - a new entanglement test based on uncertainty relations

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Composed systems & entangled pure states

K-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \ldots \otimes \mathcal{H}_K$

- **separable pure states:** $|\psi_{sep}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \otimes \ldots \otimes |\phi_K\rangle$
- **entangled pure states:** all states not of the product form.

Bi-partite systems: $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Entanglement entropy of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := \ -\mathrm{Tr}\, \sigma \ln \sigma, \qquad \sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi$$

The more mixed partial trace, the more entangled initial pure state...

Schmidt decomposition $|\psi\rangle = \sum_{m=1}^{N} \sqrt{\lambda_m} |m\rangle \otimes |m\rangle$

Entanglement measures - functions of the **Schmidt vector**

Example: Concurrence $C = \sqrt{2(1 - \sum_m \lambda_m^2)}$ (a function of purity)

Multipartite systems - geometric measure of entanglement

Wei and **Goldbart,** Phys. Rev. A **68,** 042307 (2003)

$$G(\psi) = 1 - \max_{\chi_{\rm sep}} |\langle \psi | \chi_{\rm sep} \rangle|^2$$

a 'distance' to the closest separable state of the form:

$$|\chi_{\rm sep}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \otimes \ldots \otimes |\phi_K\rangle$$

- If $|\psi\rangle$ is separable then we take $|\chi_{\rm sep}\rangle = |\psi\rangle$ and obtain $G(\psi) = 0$
- Otherwise $|\langle \psi | \chi_{\rm sep} \rangle|^2 < 1$ and $G(\psi) > 0$

Let's keep in mind this approach!

Mixed states

Definition of Entanglement

- **separable mixed states:** $\rho_{\text{sep}} = \sum_j p_j \, \rho_j^A \otimes \rho_j^B$
- **entangled mixed states:** all states not of the above form.

Entanglement measures for mixed states:

A generalization of a measure M from pure states to mixed states:

$$M(\rho) := \min_{\mathscr{E}} \sum_i p_i M(|\psi_i\rangle), \qquad \text{(convex roof)}$$

where **ensemble** $\mathscr{E} = \{p_i, |\psi_i\rangle\}$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Examples:

Entropy of formation, $E(\rho) = ;\min_{\mathscr{E}} \sum_i p_i E(|\psi_i\rangle)$
Concurrence of formation, $C(\rho) = \min_{\mathscr{E}} \sum_i p_i C(|\psi_i\rangle)$

Measurable entanglement measures

Horodecki and Ekert, Phys. Rev. Lett. 89, 127902 (2002)

measuring purity $\mathrm{Tr}\rho^2$ – two copies in a coincidence experiment
higher moments: $\mathrm{Tr}\rho^k$ – k copies in a coincidence experiment

Measurable bounds for concurrence and other stuff

- Purity difference $C(\rho) \geq \mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_A^2$ and $C(\rho) \geq \mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_B^2$ where $\rho_A = \mathrm{Tr}_{\mathrm{B}}\rho$ and $\rho_B = \mathrm{Tr}_{\mathrm{A}}\rho$ are partial traces. [**Mintert** and **Buchleitner**, Phys. Rev. Lett. **98**, 140505 (2007)]
- Generalization for other maps [**Augusiak** and **Lewenstein**, Quantum Inf. Process. **8**, 493 (2009)]
- Experimental realizations [**Walborn** et al., Nature **440**, 1022 (2006)]

Motivation

1. There is a necessity of a sensitive entanglement test (measure) for multipartite systems.
2. The test which will be suitable for dimensions larger than $N = 2$.
3. Test which could be experimentally accessible and can be generalized to the case of the mixed states.

Entropic uncertainty relations

State $|\phi\rangle = \sum_i^N a_i|\alpha_i\rangle = \sum_j^N b_j|\beta_j\rangle$ can be expanded in the eigenbases of operators $\hat{A}$ and $\hat{B}$, related by a unitary matrix $U_{ij} = \langle\alpha_i|\beta_j\rangle$. Let Shannon entropies in both expansion be:

$$S^A(\psi) = -\sum_{i=1}^{N} |a_i|^2 \ln |a_i|^2, \quad S^B(\psi) = -\sum_{j=1}^{N} |b_j|^2 \ln |b_j|^2$$

We have the following uncertainty relations ($C = \max_{ij} |U_{ij}|$):

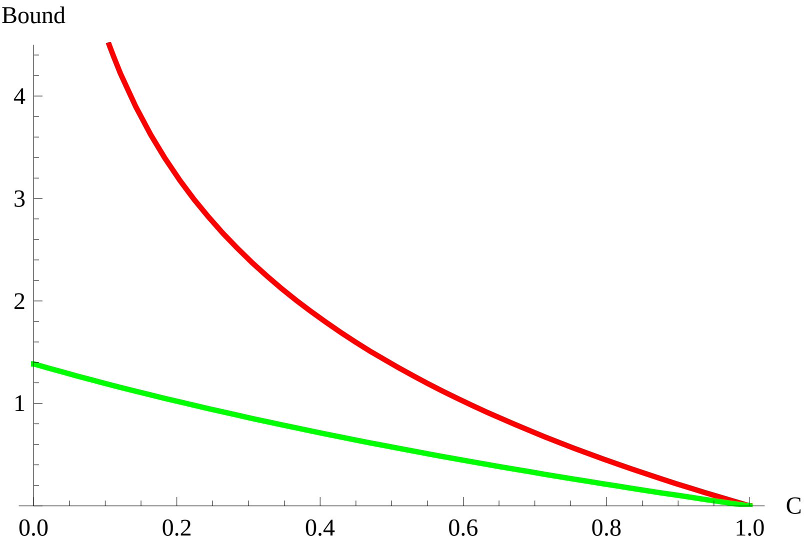
Deutsch, Phys. Rev. Lett. **50**, 631 (1983)

$$S^A(\psi) + S^B(\psi) \geq 2\ln 2 - 2\ln(1 + C)$$

Maassen and **Uffink**, Phys. Rev. Lett. **60**, 1103 (1988)

$$S^A(\psi) + S^B(\psi) \geq -2\ln C$$

Deutsch and Maassen-Uffink results



Deutsch's derivation

$$S^A(\psi)+S^B(\psi)\geq -\sum_{i=1}^{N}\sum_{j=1}^{N}|a_ib_j|^2\ln|a_ib_j|^2\geq -\ln\left(\max_{ij}|a_ib_j|^2\right)$$

In order to find the maximum one shall use a variational calculus (solve the following equation):

$$\frac{\delta}{\delta|\phi\rangle}\left(|\langle\phi|\alpha_i\rangle|^2\,|\langle\phi|\beta_j\rangle|^2-\lambda\langle\phi|\phi\rangle\right)$$

and find the „maximal" state. The solution exists and lays in a subspace spanned by $|\alpha_i\rangle$ and $|\beta_j\rangle$. **Conclusions:**

- The Deutsch upper bound $|a_ib_j|^2\leq(1+|\langle\alpha_i|\beta_j\rangle|)^2/4$ is optimal!
- The summations present in both entropies make this think worse!

A generalization

The Deutsch derivation can be generalized. Assume that $|\phi\rangle \in \mathscr{H}$ and $\dim \mathscr{H} = N$. Now, select N orthonormal vectors:

$$|\chi_j\rangle \in \mathscr{H}, \qquad j = 1, \ldots, N, \qquad \langle\chi_i|\chi_j\rangle = \delta_{ij}$$

Repeating the Deutsch's calculation we can prove that

$$\prod_{j=1}^{N} |\langle\phi|\chi_j\rangle|^2 \leq N^{-N}$$

It the previous case $N = 2$ and assuming the orthogonality $\langle\alpha_i|\beta_j\rangle = 0$ we recover the previous results!

A collective entanglement measure

Maximal collectibility for a $K-$partite pure state

Let $|\Psi\rangle \in \mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B \otimes \ldots \otimes \mathscr{H}_K$, where all dimensions are equal, $\dim(\mathscr{H}_J) = N$. Select N separable pure states, $|\chi_j^{sep}\rangle = |a_j^A\rangle \otimes \ldots \otimes |a_j^K\rangle$, where $|a_j^J\rangle \in \mathscr{H}_J$ with $j = 1, \ldots, N$ and $J = A, \ldots, K$. Now assume the local states to be **mutually orthogonal**, $\langle a_j^J | a_k^J \rangle = \delta_{jk}$. Define the **maximal collectibility [*see the preprint* arXiv:1106.2018]**

$$Y^{\max}[|\Psi\rangle] \; := \; \max_{|\chi^{sep}\rangle} \prod_{j=1}^{N} |\langle \Psi | \chi_j^{sep} \rangle|^2$$

- In „Geometric measure of entanglement” $G(\Psi)$ the state is highly **entangled** state when $|\langle\Psi|\chi^{sep}\rangle|^2$ is the smallest.
- In the collectibility approach the state is highly **entangled** when $|\langle\Psi|\chi_1^{sep}\rangle|^2 \cdots |\langle\Psi|\chi_N^{sep}\rangle|^2$ is the **largest**!

An Upper bounds for the collectibility

For any pure state $|\Psi\rangle$ we have the following bound

$$Y^{\max}\left[|\Psi\rangle\right] \;\leq\; N^{-N},$$

Which is saturated by the maximally entangled state,
$|\Psi_+\rangle = \frac{1}{\sqrt{N}}\sum_i |i,i\rangle$ (bi-partite case) and a generalized GHZ state
$|GHZ\rangle_K = \frac{1}{\sqrt{N}}\sum_i |i\rangle_A \otimes \cdots \otimes |i\rangle_K$ in a K–partite case.

But if the state is **separable,** $|\Psi_{sep}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \otimes \ldots \otimes |\Psi_K\rangle$, then

$$\prod_{j=1}^{N} |\langle \Psi_{sep}|\chi_j^{sep}\rangle|^2 = \prod_{l=1}^{K}\prod_{j=1}^{N}\left|\left\langle \Psi_l | a_j^l \right\rangle\right|^2 \leq N^{-N\cdot K}$$

A separability criterion

For any separable state the following separability criterion works

$$Y^{max}[|\Psi_{\rm sep}\rangle] \;\leq\; N^{-N\cdot K} \quad\Longrightarrow\quad \text{A \textbf{separability} criterion}$$

$Y^{max}[|\Psi\rangle] \geq N^{-N\cdot K}$ then the state $|\Psi\rangle$ is **entangled**

the state	N^{-N}	$N^{-N\cdot K}$
two qubits	$1/4 = 0.25000$	$1/16 = 0.06250$
three qubits	$1/4 = 0.25000$	$1/64 \approx 0.01563$
two qutrits	$1/27 \approx 0.03704$	$1/729 \approx 0.00137$

A partial collectibility for a general $N^{\otimes}K$ system

To find $Y^{\max}[|\Psi\rangle]$ we need to optimize over a base consisting of N separable states $|\chi_j^{sep}\rangle$! Start with a **single** optimization over the subspace $\mathscr{H}^A$, and define the **partial collectibility**:

$$Y_a[|\Psi\rangle] := \max_{|a^A\rangle} \prod_{j=1}^{N} |\langle\Psi|\chi_j^{sep}\rangle|^2$$

parametrized by the set a of N product states $|a_j^B\rangle \otimes \ldots \otimes |a_j^K\rangle$.

Collectibility for K qubit system ($N=2$)

$$Y_a\left[|\Psi\rangle\right] = \frac{1}{4}\left(\sqrt{G_{11}G_{22}} + \sqrt{G_{11}G_{22} - |G_{12}|^2}\right)^2,$$

where $G_{jk} = \langle\varphi_j|\varphi_k\rangle$ is a **Gram matrix** among projected states, so that $|\varphi_j\rangle = \left(\left\langle a_j^B\right| |\otimes \ldots \otimes \left\langle a_j^K\right|\right) |\Psi_{AB}\rangle \in \mathscr{H}^A$.

Collectibility for two qubit system, $K = 2$ and $N = 2$

We write a bi-partite pure state in its **Schmidt form**
$|\Psi_{AB}\rangle = (U_A \otimes U_B)\left[\cos\left(\frac{\psi}{2}\right)|00\rangle + \sin\left(\frac{\psi}{2}\right)|11\rangle\right]$
Direct optimization gives its **maximal collectibility**

$$Y^{max}(\psi) = \left(\frac{1+\sin(\psi)}{4}\right)^2$$

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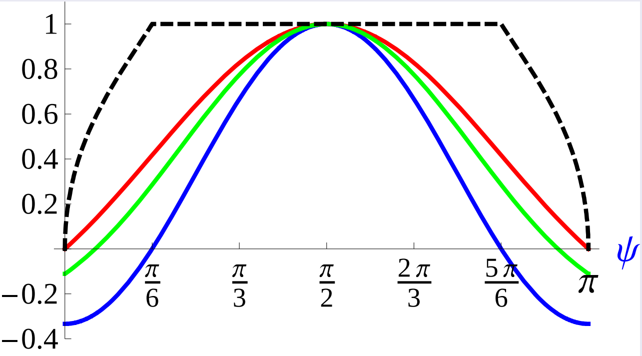
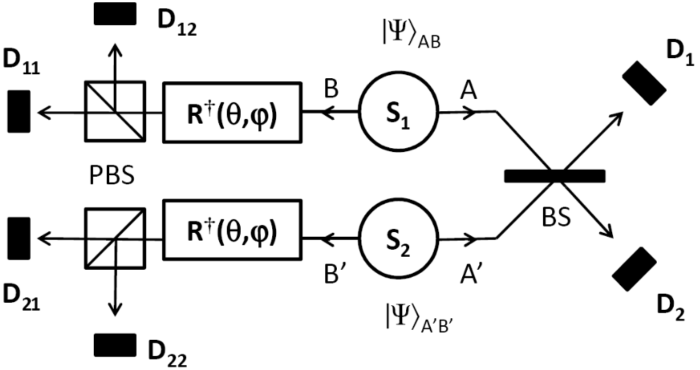


On the **left side** „B" the statistics of pairs of clicks after projections onto detectors are measured.

On the **right side** „A" the **Hong–Ou–Mandel interference** is performed.



The number $|G_{12}|^2$ is equal to the probability of the pair of the **clicks at „B"** **multiplied by that** **of a double click at „A"**.

Collectibility for three qubit system, $K = 3$ and $N = 2$

The **collectibility** is maximal for a

a) GHZ state, $|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and then $Y^{\max}[|GHZ\rangle] = 16/64 = 1/4$, while for

b) W–state, $|W\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ it reads $Y^{\max}[|W\rangle] = 9/64$. For a

c) Bi–separable state, $|BS\rangle = |\Psi\rangle_{AB} \otimes |\phi\rangle_C$ one has $Y^{\max}[|BS\rangle] = 4/64 = 1/16$, while for

d) Separable state the **collectibility** reads, $Y^{\max}[|\Psi_{\mathrm{sep}}\rangle] = 1/64$.

Collectibility as a detector of the **genuine entanglement**

Thus any measured value of $Y^{\max}[|\Psi\rangle]$ above $1/16$ provides an evidence for **genuine three–party entanglement** for the analyzed state $|\Psi\rangle$!

Concluding remarks

1. We introduced **collectibility** $Y^{\max}(\Psi)$ **as a function of any pure state** $|\Psi\rangle$ **of a composed** $N^{\otimes K}$ **system,**
2. **Collectibility** satisfies inequalities analogous to **entropic uncertainty relations**
3. The **partial collectibility** $Y_a(\Psi)$ labeled by parameters describing positions of detectors $|a_j^B\rangle,\ldots,|a_j^K\rangle$ is experimentally accessible for any K–qubit system,
4. An experimental photonic scheme based on **Hong–Ou–Mandel interferometry** is proposed to measure **collectibility** in a two-qubit system.
5. Results for **collectibility**, presented here for **pure states** only, can also be generalized for **non-ideal pure states** with **purity** less then one, $\mathrm{Tr}\rho^2 > 1-\varepsilon$.

accepted to PRL (coming soon), see the preprint
arXiv:1106.2018