PHASE SPACE THEORY OF BOSE-EINSTEIN CONDENSATES

AND

TIME-DEPENDENT MODES

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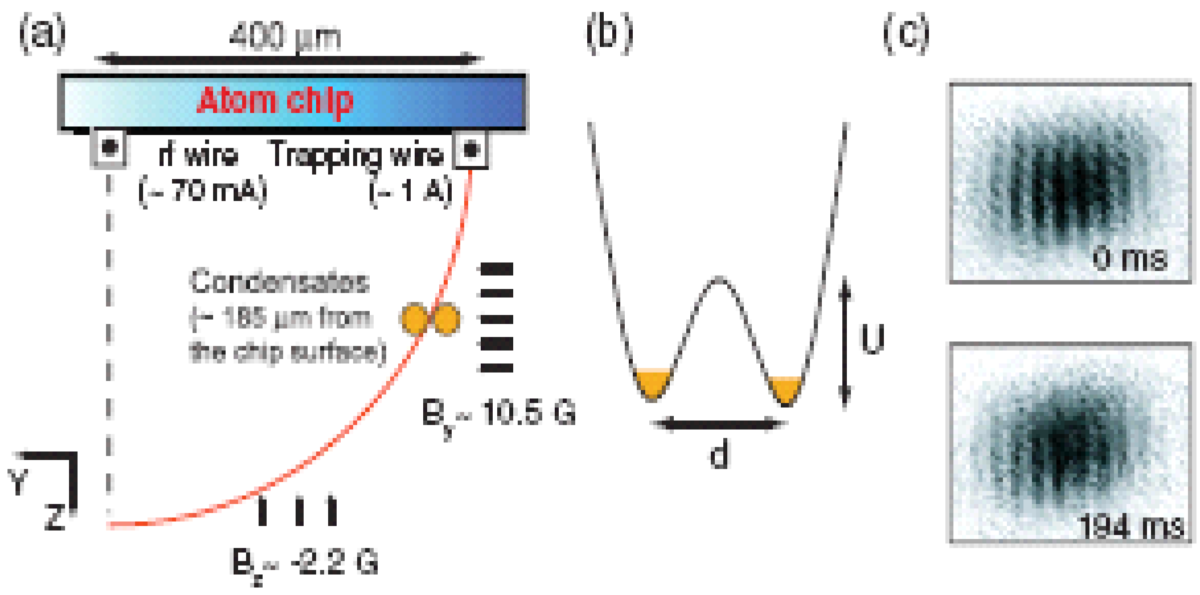
TOPIC

♦ General theory of BEC interferometry

- Treat *two mode* cases such as *one-component* BEC in *double wells*.
- Theory based on *mean field* and *phase space* methods.
- Include *dephasing* and *decoherence*.
- Obtain *quantum correlation functions*.
- *Typical* BEC interferometry experiment



Jo et al, Phys Rev Letts 98, 030407 (2007)

♦ Present work

- *One-component* BEC in *double well*.
- Mean field theory based on *Dirac-Frenkel variational principle* for *two-mode* quantum state.
- Phase space theory based on *hybrid Wigner*, *P+ distribution functional*.
- New terms in *functional Fokker-Planck* and *Ito stochastic field* equations due to *time dependent mode* functions.
- Extends previous work - B J Dalton; Annals of Physics **326**, 668 (2011).

♦ Future work

- *Numerical studies* based on mean field and phase space theory.
- *Develop general theory* for other two mode cases such as *two component* BEC in a *single well*.

MOTIVATION

♦ Bose-Einstein condensates in cold atomic gases

- All N bosons occupy *small* number of single particle states (or *modes*) – often only one mode $(T \ll T_c)$.
- Quantum system on a *macroscopic* scale $N \gg 1$ with *massive* particles $\lambda_{compton} \sim 10^{-30} m$.
- Long range *spatial coherence*.
- Controllable *experiments* - trap potentials, Feshbach resonances, one and two component BEC, 1D and 2D BEC,..
- Ideal for studying *quantum interferometry*, *decoherence*, *entanglement* in a *macroscopic* system of *localisable* bosons.
- Suitable system for *precision measurements*.

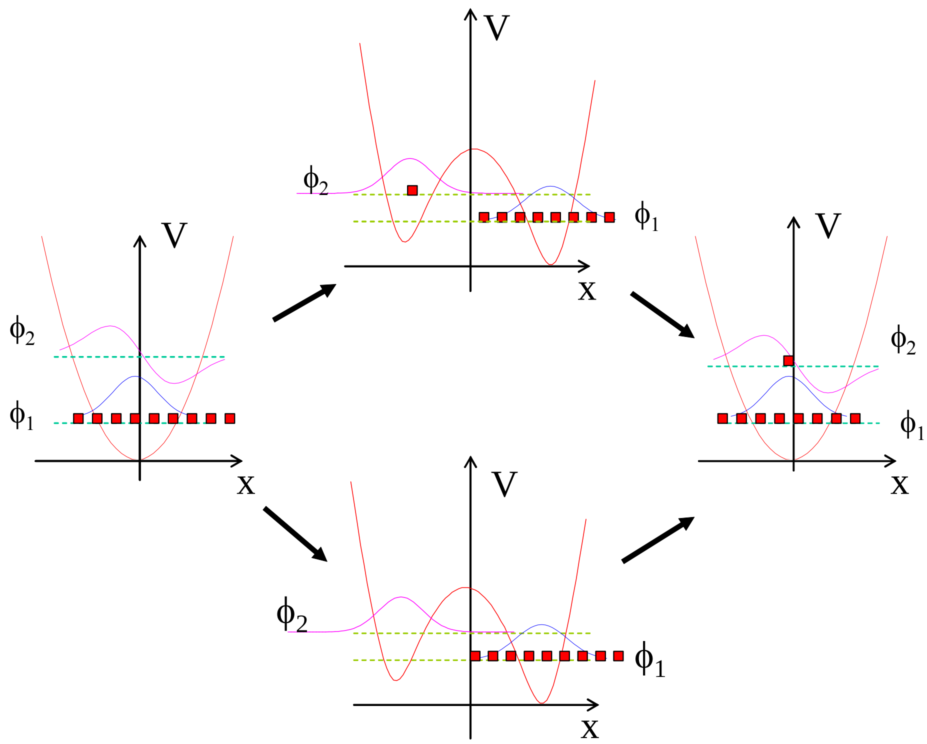
♦ BEC interferometry

- Based on *almost all* bosons in one (or two) modes.
- Involves all topics - *QInterf, Decoh, PrecM, Entang*.
- Many types - *Ramsey* interferometry, *Mach-Zender*, *Bragg*, ..
- Description - *quantum correlation functions* - expectation values of products of bosonic field operators - related to many-boson *position measurements*.

♦ Quantum interference

- Mach-Zender *double-well interferometry experiment* shown.
- Essentially a *two-mode* case.
- Involves starting with BEC in single well trap, changing trap to (possibly *asymmetric*) double-well trap and back to single well.

• Asymmetry could lead to *excitation* of some bosons to higher energy states of final trap (shown), or to changes to *spatial* interference patterns (not shown).



• *Process* of one boson excitation shown with two *quantum pathways*, both involving intermediate double well trap.

• *Near degeneracy* of energy levels for asymmetric double well facilitates boson transfer to excited state.

• *Superposition* of *transition amplitudes* gives *quantum interference* effects.

♦ Decoherence

- If boson-boson *interactions* were absent and BEC isolated, QCF result in *clearly visible* interferometric effects.
- *Internal* boson-boson interactions result in *dephasing* (due to transitions *within* condensate modes) and *decoherence* effects (due to transitions *from* condensate modes) that *degrade* interference pattern.

♦ Precision measurement

- BEC interferometry offers possible *precision improvements* by a factor given by $\sqrt{N}$ (Kasevich (2002); Dunningham, Barnett, Burnett (2002)) - *Heisenberg limit*.

♦ Entanglement

- Entangled and non-entangled states lead to *differing* BEC interferometry effects.

SINGLE COMPONENT BEC

♦ Hamiltonian

$$\hat{H} = \int dr\left(\frac{\hbar^2}{2m}\nabla\hat{\Psi}(r)^{\dagger}.\nabla\hat{\Psi}(r) + \hat{\Psi}(r)^{\dagger}V\hat{\Psi}(r)\right.$$
$$\left. + \frac{g}{2}\hat{\Psi}(r)^{\dagger}\hat{\Psi}(r)^{\dagger}\hat{\Psi}(r)\hat{\Psi}(r)\right)$$

♦ Field operators

$$[\hat{\Psi}(r), \hat{\Psi}^{\dagger}(s)] = \delta(r - s)$$

♦ Quantum correlation functions

$$G^n(r_1..r_p; s_q..s_1)$$
$$= \langle\hat{\Psi}(r_1)^{\dagger}..\hat{\Psi}(r_p)^{\dagger}\hat{\Psi}(s_q)..\hat{\Psi}(s_1)\rangle$$

♦ Mode expansion

$$\hat{\Psi}(x) = \sum_k \hat{a}_k(t)\phi_k(x,t) \quad \hat{\Psi}^{\dagger}(x) = \sum_k \hat{a}_k^{\dagger}(t)\phi_k^*(x,t)$$

- Fields time *independent*, modes time *dependent*.

♦ Mode annihilation, creation operators

$$[\hat{a}_k(t), \hat{a}_l^\dagger(t)] = \delta_{kl}$$

♦ Mode orthonormality, completeness

$$\int dx\,\phi_k^*(x,t)\phi_l(x,t) = \delta_{kl} \qquad \sum_k \phi_k(x,t)\phi_k^*(y,t) = \delta(x-y)$$

♦ Mode time dependency

$$\frac{\partial \hat{a}_k(t)}{\partial t} = \sum_l C_{kl}(t)\hat{a}_l(t)$$

$$\frac{\partial \phi_k(x,t)}{\partial t} = \sum_l C_{kl}^*(t)\phi_l(x,t)$$

♦ Coupling constants

$$C_{kl}(t) = \int dx \frac{\partial \phi_k^*(x,t)}{\partial t}\phi_l(x,t) = iD_{kl}(t)$$

$$C_{kl} + C_{lk}^* = 0$$

- The coupling constants play a *key role* in the theory.

HYBRID MODEL

♦ Physics of BEC well below T_c

• Most bosons occupy one or two *condensate* modes - describe via *mean field theory* based on generalised *Gross-Pitaevskii equations*.

• Treat condensate modes via *Wigner distribution function*.

• Few bosons occupy *non-condensate* modes - *quantum effects*.

• Treat non-condensate modes via *Positive P distribution function*.

• Details: Dalton, ArXiv Cond-Matt 1108.1251.

• References: Steel et al, PRA **58**, 4824 (1998); Gardiner et al, PRA **58**, 536 (1998); Dalton, J Phys C Conf Ser **67**, 012059 (2007); Hoffmann et al, PRA **78**, 013622 (2008); Dalton, Ann Phys **326**, 668 (2011).

♦ Condensate modes

• Based on Dirac-Frenkel *variational principle*.

- Minimise *dynamical action*

$$S_{dyn} = \int dt \left(\begin{array}{c} \{\langle \partial_t \Phi | \Phi \rangle - \langle \Phi | \partial_t \Phi \rangle\} / 2i \\ -\left\langle \Phi \right| \widehat{H} \left| \Phi \right\rangle / \hbar \end{array} \right)$$

♦ Two mode quantum state

- Superposition of $N+1$ *basis states* $|\frac{N}{2}, k\rangle$, where $\frac{N}{2}-k$ and $\frac{N}{2}+k$ bosons occupy two modes with *mode functions* ϕ_1 and ϕ_2. $(k = -N/2, -N/2+1, \ldots, +N/2)$.

$$\left|\Phi(t)\right\rangle = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} b_k(t) \left| \frac{N}{2}, k \right\rangle.$$

- Basis states are *Fock states* - these states are *fragmented*

$$\left| \frac{N}{2}, k \right\rangle = \frac{(\hat{a}_1(t)^\dagger)^{(\frac{N}{2}-k)}}{[(\frac{N}{2}-k)!]^{\frac{1}{2}}} \frac{(\hat{a}_2(t)^\dagger)^{(\frac{N}{2}+k)}}{[(\frac{N}{2}+k)!]^{\frac{1}{2}}} |0\rangle$$

- Basis state *amplitudes* are $b_k(t)$.
- *Generalised Gross-Pitaevskii eqns* for modes and *matrix eqns* for amplitudes are *coupled* and *self-consistent*.

PHASE SPACE THEORY

♦ Basic idea - separate modes

- Mode annihiln, creation oprs $\hat{a}_k(t)$, $\hat{a}_k^\dagger(t)$ represented by *phase space variables* $\alpha_k(t)$, $\alpha_k^+(t)$.
- Density operator $\hat{\rho}(t)$ represented by *distribution functions* $P(\underset{\to}{\alpha}, \underset{\to}{\alpha}^*, t)$ or $W(\underset{\to}{\alpha}, \underset{\to}{\alpha}^*, t)$ with $\underset{\to}{\alpha} \equiv \{\alpha_k, \alpha_k^+\}$.
- QCF $\to$ *phase space averages.*
- *Normally ordered* QCF

$$G(l_1, l_2, ..l_p; m_q, .., m_2, m_1)$$

$$= \mathrm{Tr}(\hat{\rho}\,\hat{a}_{l_1}^\dagger\,\hat{a}_{l_2}^\dagger..\hat{a}_{l_n}^\dagger\,\hat{a}_{m_n}..\,\hat{a}_{m_2}\hat{a}_{m_1})$$

$$= \int\int d^2\alpha\, d^2\alpha^+\, \alpha_{l_1}^+..\alpha_{l_p}^+\, \alpha_{m_q}..\alpha_{m_1}\, P(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$$

- Phase space *integration*: $\alpha_k = \alpha_{kx} + i\alpha_{ky}$

$$\int\int d^2\alpha(t)\, d^2\alpha^+(t) \equiv \int\int \prod_k d\alpha_{kx}\, d\alpha_{ky} \prod_k d\alpha_{kx}^+\, d\alpha_{ky}^+$$

♦ Basic idea - fields

- Field annihiln, creation oprs $\hat{\Psi}(x)$, $\hat{\Psi}^{\dagger}(x)$ represented by *field functions* $\psi(x)$, $\psi^{+}(x)$.
- Density operator $\hat{\rho}(t)$ represented by *distribution functionals* $P[\underrightarrow{\psi},\underrightarrow{\psi}^{*},t]$ or $W[\underrightarrow{\psi},\underrightarrow{\psi}^{*},t]$ with $\underrightarrow{\psi}\equiv\{\psi,\psi^{+}\}$.
- QCF $\rightarrow$ *functional integral averages*.
- *Symmetrically ordered* QCF

$$
\begin{aligned}
&G^{W}(r_1\cdots r_p;s_q\cdots s_1)\\
&=\mathrm{Tr}(\hat{\rho}\,\{\hat{\Psi}(r_1)^{\dagger}\cdots\hat{\Psi}(r_p)^{\dagger}\hat{\Psi}(s_q)\cdots\hat{\Psi}(s_1)\})\\
&=\int\!\!\int D^2\psi\, D^2\psi^{+}\,\psi^{+}(r_1)..\psi^{+}(r_p)\psi(s_q)..\psi(s_1)\\
&\quad\times W[\psi,\psi^{+},\psi^{*},\psi^{+*}]
\end{aligned}
$$

- Symmetrically ordered is *average* of the *products* of operators in *all orders*.

♦ Modes and fields equivalence

- *Fnal*, *phase space* integn equivalent.

$$
\int\!\!\int D^2\psi\, D^2\psi^{+}\,F[\underrightarrow{\psi},\underrightarrow{\psi}^{*}] \;\equiv\int\!\!\int d^2\alpha(t)\,d^2\alpha^{+}(t)\,f(\underrightarrow{\alpha},\underrightarrow{\alpha}^{*})
$$

- $F[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$ *equivalent* to $f(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*)$.
- Field *expansion*

$$\psi(x) = \sum_k \alpha_k(t)\phi_k(x,t) \quad \psi^+(x) = \sum_k \alpha_k^+(t)\phi_k^*(x,t)$$

- For *n modes*, grid of *n spatial intervals*.

♦ Key step: phase variable evoln

- Choose *same* as for mode operators

$$\frac{\partial \alpha_k(t)}{\partial t} = \sum_l C_{kl}(t)\alpha_l(t) \quad \frac{\partial \alpha_k^+(t)}{\partial t} = \sum_l C_{kl}^*(t)\alpha_l^+(t)$$

- *Field fns* $\psi(x)$, $\psi^+(x)$ time *independent*.
- *Formal solution* involves *unitary* matrix

$$\alpha_k(t) = \sum_l U_{kl}(t)\alpha_l(0) \quad \alpha_k^+(t) = \sum_l U_{kl}^*(t)\alpha_l^+(0)$$

$$\frac{\partial U_{kl}(t)}{\partial t} = i\sum_m D_{km}(t)U_{ml}(t)$$

- Phase space, fnal intn *time independent*

$$\int\int d^2\alpha(t)d^2\alpha^+(t) \equiv \int\int d^2\alpha(0)d^2\alpha^+(0)$$

HYBRID DISTRIBUTION FUNCTIONAL

- **Condensate, non-condensate field oprs**

$$\hat{\Psi}_C(\mathrm{x},\mathrm{t}) = \sum_{k\epsilon C} \hat{a}_k(\mathrm{t})\phi_k(\mathrm{x},\mathrm{t}) \quad \hat{\Psi}_C^\dagger(\mathrm{x},\mathrm{t}) = \sum_{k\epsilon C} \hat{a}_k^\dagger(\mathrm{t})\phi_k^*(\mathrm{x},\mathrm{t})$$

$$\hat{\Psi}_{NC}(\mathrm{x},\mathrm{t}) = \sum_{k\epsilon NC} \hat{a}_k(\mathrm{t})\phi_k(\mathrm{x},\mathrm{t}) \quad \hat{\Psi}_{NC}^\dagger(\mathrm{x},\mathrm{t}) = \sum_{k\epsilon NC} \hat{a}_k^\dagger(\mathrm{t})\phi_k^*(\mathrm{x},\mathrm{t})$$

- Mode sums *restricted* to condensate or non-condensate modes.
- *Sum* gives time *independent* total field operators $\hat{\Psi} = \hat{\Psi}_C + \hat{\Psi}_{NC}$, $\hat{\Psi}^\dagger = \hat{\Psi}_C^\dagger + \hat{\Psi}_{NC}^\dagger$.
- Separate field oprs time *dependent*.

♦ Characteristic functional

$$\chi[\underrightarrow{\Xi}]=\mathrm{Tr}(\hat{\Omega}^{W}[\Xi_{C},\Xi_{C}^{+}]\,\hat{\Omega}^{+}[\Xi_{NC}^{+}]\,\hat{\rho}\,\hat{\Omega}^{-}[\Xi_{NC}])$$

$$\hat{\Omega}^{+}[\Xi_{NC}^{+}]=\exp i\int dx\,\hat{\Psi}_{NC}(x,t)\,\Xi_{NC}^{+}(x,t)$$

$$\hat{\Omega}^{-}[\Xi_{NC}]=\exp i\int dx\,\Xi_{NC}(x,t)\,\hat{\Psi}_{NC}^{\dagger}(x,t)$$

$$\hat{\Omega}^{W}[\Xi_{C},\Xi_{C}^{+}]=\exp i\int dx\,(\hat{\Psi}_{C}(x,t)\,\Xi_{C}^{+}(x,t)+\Xi_{C}(x,t)\,\hat{\Psi}_{C}^{\dagger}(x,t))$$

where $\underrightarrow{\Xi} \equiv \{\Xi_{C},\Xi_{C}^{+},\Xi_{NC},\Xi_{NC}^{+}\}$

- *Baker-Haussdorff* theorem gives

$$\chi[\underrightarrow{\Xi}]=\exp\left\{-\frac{1}{2}\int dx\,\Xi_{C}(x,t)\,\Xi_{C}^{+}(x,t))\right\}\chi_{P+}[\underrightarrow{\Xi}]$$

$$\chi_{P+}[\underrightarrow{\Xi}]=\mathrm{Tr}(\hat{\Omega}^{+}[\Xi_{C}^{+}]\,\hat{\Omega}^{+}[\Xi_{NC}^{+}]\,\hat{\rho}\,\hat{\Omega}^{-}[\Xi_{NC}]\,\hat{\Omega}^{-}[\Xi_{C}])$$

- Relates *hybrid* and *normally ordered* characteristic functionals.

♦ Characteristic functional fields

$$\Xi_{C}(x,t)=\sum_{k\epsilon C}\xi_{k}(t)\phi_{k}(x,t)\qquad \Xi_{C}^{+}(x,t)=\sum_{k\epsilon C}\xi_{k}^{+}(t)\phi_{k}^{*}(x,t)$$

$$\Xi_{NC}(x,t)=\sum_{k\epsilon NC}\xi_{k}(t)\phi_{k}(x,t)\qquad \Xi_{NC}^{+}(x,t)=\sum_{k\epsilon NC}\xi_{k}^{+}(t)\phi_{k}^{*}(x,t)$$

- *Sum* gives time *independent* total fns $\Xi = \Xi_C + \Xi_{NC}$.
- Separate fns time *dependent*.

♦ Characteristic variable evolution

- Choose *same* as mode oprs

$$\frac{\partial \xi_k(t)}{\partial t} = \sum_l C_{kl}(t)\xi_l(t) \qquad \frac{\partial \xi_k^+(t)}{\partial t} = \sum_l C_{kl}^*(t)\xi_l^+(t)$$

♦ Condensate, non-condensate fields

$$\psi_C(x,t) = \sum_{k\epsilon C} \alpha_k(t)\phi_k(x,t) \qquad \psi_C^+(x,t) = \sum_{k\epsilon C} \alpha_k^+(t)\phi_k^*(x,t)$$

$$\psi_{NC}(x,t) = \sum_{k\epsilon NC} \alpha_k(t)\phi_k(x,t) \qquad \psi_{NC}^+(x,t) = \sum_{k\epsilon NC} \alpha_k^+(t)\phi_k^*(x,t)$$

- *Sum* gives time *independent* total field fns $\psi = \psi_C + \psi_{NC}$, $\psi^+ = \psi_C^+ + \psi_{NC}^+$.
- Separate fns time *dependent*.

♦ Distribution functional

$$\chi[\underset{\rightarrow}{\Xi}]$$

$$= \int\int\int\int D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+$$

$$\times \exp\left(i\int dx\,(\psi(x)\,\Xi^+(x) + \Xi(x)\,\psi^+(x)\right)$$

$$\times P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$$

- Distribution functional *not unique* or *analtyic*.
- Functional integration

$$\underset{\rightarrow}{\psi} \equiv \{\psi_C, \psi_C^+, \psi_{NC}, \psi_{NC}^+\}$$

$$\int\int D^2\psi \, D^2\psi^+ \, F[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$$

$$\equiv \int\int\int\int D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, F[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$$

- *Quantum correlation functions* given as *functional integrals*.

FUNCTIONAL FOKKER-PLANCK EQUATION

- **Key step**

$$\frac{\partial}{\partial t}\chi[\underrightarrow{\Xi};\hat{\rho}]$$

$$= \int D^2\underrightarrow{\psi}\,\exp\,(i\int dx\,(\psi(x)\,\Xi^+(x) + \Xi(x)\,\psi^+(x))\,\frac{\partial}{\partial t}P[\underrightarrow{\psi},\underrightarrow{\psi}^*]$$

$$= \chi[\underrightarrow{\Xi};\frac{\partial}{\partial t}\hat{\rho}] + \frac{\partial}{\partial t}\left\{-\frac{1}{2}\int dx\,\Xi_C(x,t)\,\Xi_C^+(x,t))\right\}\chi[\underrightarrow{\Xi};\hat{\rho}]$$

- 1st term gives *usual* FFPE terms.
- 2nd term gives extra *diffusion* terms.

$$-\frac{\partial}{\partial t}\int dx\,\Xi_C(x,t)\,\Xi_C^+(x,t) =$$

$$\int\int dx\,dy\,\{\sum_{k\epsilon C}\sum_{l\epsilon NC}\phi_l^*(x,t)C_{kl}(t)\phi_k(y,t)\}(i\Xi_{NC}(x,t))(i\Xi_C^+(y,t))$$

$$+\int\int dx\,dy\,\{\sum_{l\epsilon C}\sum_{k\epsilon NC}\phi_l^*(x,t)C_{lk}^*(t)\phi_k(y,t)\}(i\Xi_C(x,t))(i\Xi_{NC}^+(y,$$

- Derive FFPE via *correspondence rules*.

♦ Correspondence rules

$$\hat{\rho} \Rightarrow \hat{\Psi}_{NC}(x,t)\,\hat{\rho} \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \psi_{NC}(x,t)\,P$$

$$\hat{\rho} \Rightarrow \hat{\rho}\,\hat{\Psi}_{NC}(x,t) \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(-\frac{\delta}{\delta\psi^+_{NC}(x,t)} + \psi_{NC}(x,t)\right)P$$

$$\hat{\rho} \Rightarrow \hat{\Psi}^\dagger_{NC}(x,t)\,\hat{\rho} \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(-\frac{\delta}{\delta\psi_{NC}(x,t)} + \psi^+_{NC}(x,t)\right)P$$

$$\hat{\rho} \Rightarrow \hat{\rho}\,\hat{\Psi}^\dagger_{NC}(x,t) \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \psi^+_{NC}(x,t)P$$

$$\hat{\rho} \Rightarrow \hat{\Psi}_{C}(x,t)\hat{\rho} \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(\psi_C(x,t) + \frac{1}{2}\frac{\delta}{\delta\psi^+_C(x,t)}\right)P$$

$$\hat{\rho} \Rightarrow \hat{\rho}\hat{\Psi}_{C}(x,t) \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(\psi_C(x,t) - \frac{1}{2}\frac{\delta}{\delta\psi^+_C(x,t)}\right)P$$

$$\hat{\rho} \Rightarrow \hat{\Psi}^\dagger_{C}(x,t)\hat{\rho} \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(\psi^+_C(x,t) - \frac{1}{2}\frac{\delta}{\delta\psi_C(x,t)}\right)P$$

$$\hat{\rho} \Rightarrow \hat{\rho}\hat{\Psi}^\dagger_{C}(x,t) \qquad P[\underset{\to}{\psi},\underset{\to}{\psi}^*] \Rightarrow \left(\psi^+_C(x,t) + \frac{1}{2}\frac{\delta}{\delta\psi_C(x,t)}\right)P$$

- FFPE written in *new notation*.

♦ Notation change

- Field fns: $\psi_C, \psi_C^+, \psi_{NC}, \psi_{NC}^+ \to \psi_A^\mu$ (where $\mathrm{A} = \mathrm{C}, \mathrm{NC}$ and $\mu = -, +$; $\psi_A^- = \psi_A, \psi_A^+ = \psi_A^+$)
- Mode fns: $\phi_k, \phi_k^* \to \phi_{Ak}^\mu$ (where $\mathrm{A} = \mathrm{C}, \mathrm{NC}$ and $\mu = -, +$; $\phi_{Ak}^- = \phi_{Ak}, \phi_{Ak}^+ = \phi_{Ak}^*$)
- Phase variables: $\alpha_k, \alpha_k^+ \to \alpha_{\mu\ Ak}$ (where $\mathrm{A} = \mathrm{C}, \mathrm{NC}$ and $\mu = -, +$; $\alpha_{-\ Ak} = \alpha_{Ak}, \alpha_{+\ Ak} = \alpha_{Ak}^+$)
- Coupling constants: $\mathrm{c}_{kl} \to \mathrm{c}_{Ak\ Bl}^\mu$ (where $\mathrm{A} = \mathrm{C}, \mathrm{NC}$ and $\mu = -, +$; $\mathrm{c}_{\mathrm{Ak\ Bl}}^- = \mathrm{c}_{\mathrm{Ak\ Bl}}, \mathrm{c}_{\mathrm{Ak\ Bl}}^+ = \mathrm{c}_{\mathrm{Ak\ Bl}}^*$)
- Characteristic fields: $\Xi_C, \Xi_C^+, \Xi_{NC}, \Xi_{NC}^+ \to \Xi_A^\mu$
- Characteristic variables: $\xi_k, \xi_k^+ \to \xi_{\mu\ Ak}$
- Examples:

$$\psi_A^\mu(\mathrm{x}, \mathrm{t}) = \sum_k \alpha_{\mu\ Ak}(\mathrm{t})\phi_{Ak}^\mu(\mathrm{x}, \mathrm{t})$$

$$\frac{\partial \alpha_{\mu\ Ak}(\mathrm{t})}{\partial \mathrm{t}} = \sum_{Bl} \mathrm{c}_{Ak\ Bl}^\mu(\mathrm{t})\alpha_{\mu\ Bl}(\mathrm{t})$$

$$\mathrm{c}_{Ak\ Bl}^\mu(\mathrm{t}) = \int \mathrm{dx} \frac{\partial \phi_{Ak}^{-\mu}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{t}} \phi_{Bl}^\mu(\mathrm{x}, \mathrm{t})$$

♦ Functional FPE

- *KEY RESULT*

$$\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*] =$$

$$-\sum_{\mu A} \int dx \frac{\delta}{\delta \psi_A^{\mu}(x,t)} A_A^{\mu}(x)\ P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$$

$$+\frac{1}{2} \sum_{\mu A, \nu B} \int\int dx\, dy \frac{\delta}{\delta \psi_A^{\mu}(x,t)} \frac{\delta}{\delta \psi_B^{\nu}(y,t)} E_{AB}^{\mu\nu}(x,y)\ P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*]$$

- *Drift* term involving $A_A^{\mu}(x)$.
- *Diffusion* term involving $E_{AB}^{\mu\nu}(x,y)$.

$$E_{AB}^{\mu\nu}(x,y) = D_{AB}^{\mu\nu}(x,y)$$

$$+\frac{1}{2}\delta_{AC}\delta_{BNC}\delta_{\mu-\nu}\left(\sum_k \sum_l \phi_{Ak}^{\mu}(x,t) C_{AkBl}^{\mu}(t) \phi_{Bl}^{\nu}(y,t)\right)$$

$$+\frac{1}{2}\delta_{BC}\delta_{ANC}\delta_{\nu-\mu}\left(\sum_k \sum_l \phi_{Bl}^{\nu}(y,t) C_{BlAk}^{\nu}(t) \phi_{Ak}^{\mu}(x,t)\right)$$

- Hybrid fnal FPE has *same drift vector* $A_A^{\mu}(x)$ but *different diffusion matrix* $E_{AB}^{\mu\nu}(x,y)$ to diffusion matrix $D_{AB}^{\mu\nu}(x,y)$ obtained via $\frac{\partial}{\partial t}\hat{\rho}$ term.

- Extra terms involve *time dependent mode fns* and *coupling coefficients*.
- Extra terms involve *only* condensate to non-condensate mode couplings.
- Diffusion matrix *symmetric* $E_{AB}^{\mu\nu}(x,y) = E_{BA}^{\nu\mu}(y,x)$.
- FFPE terms involving *third* and *higher* order derivatives arising via $\frac{\partial}{\partial t}\hat{\rho}$ term *discarded* due to *scaling* as higher powers of $1/\sqrt{N}$.

ITO STOCHASTIC FIELD EQUATIONS

♦ Basic idea

- Replace non-stochastic fields

$\underrightarrow{\psi}=\{\psi_C,\psi_C^+,\psi_{NC},\psi_{NC}^+\}$ by *stochastic fields*

$\underrightarrow{\psi}^s=\{\psi_C^s,\psi_C^{s+},\psi_{NC}^s,\psi_{NC}^{s+}\}$

$$\psi_C^s(x,t)=\sum_{k\epsilon C}\alpha_k^s(t)\phi_k(x,t) \quad \psi_C^{s+}(x,t)=\sum_{k\epsilon C}\alpha_k^{s+}(t)\phi_k^*(x,t)$$

$$\psi_{NC}^s(x,t)=\sum_{k\epsilon NC}\alpha_k^s(t)\phi_k(x,t) \quad \psi_{NC}^{s+}(x,t)=\sum_{k\epsilon NC}\alpha_k^{s+}(t)\phi_k^*(x,t)$$

- Stochastic feature due to replacing non-stochastic phase variables α_k,α_k^+ by *stochastic phase variables* α_k^s,α_k^{s+}.
- Phase space and stochastic *average* of $F[\underrightarrow{\psi}]$ to be *same* for *arbitrary* $F[\underrightarrow{\psi}]$.

- *Phase space* fnal average $\left\langle F[\underrightarrow{\psi}]\right\rangle_t$

$$\left\langle F[\underrightarrow{\psi}]\right\rangle_t = \int D^2\underrightarrow{\psi}\, F[\underrightarrow{\psi}]\, P[\underrightarrow{\psi},\underrightarrow{\psi}^*]$$

- *Stochastic* average of M samples $\underrightarrow{\psi}_i^s$.

$$\overline{F[\underrightarrow{\psi}^s]} = \frac{1}{M}\sum_{i=1}^{M} F[\underrightarrow{\psi}_i^s]$$

♦ Key step: phase space fnal average

$$\frac{\partial}{\partial t}\left\langle F[\underrightarrow{\psi}]\right\rangle_t$$

$$= \int D^2\underrightarrow{\psi}\, F[\underrightarrow{\psi}]\frac{\partial}{\partial t}P[\underrightarrow{\psi},\underrightarrow{\psi}^*] + \int D^2\underrightarrow{\psi}\,\frac{\partial}{\partial t}F[\underrightarrow{\psi}]\, P[\underrightarrow{\psi},\underrightarrow{\psi}^*]$$

- 1st term gives *usual* terms from FFPE.
- 2nd term gives *extra* drift-like terms.

$$\frac{\partial}{\partial t}F[\underrightarrow{\psi}] = \sum_{\mu A}\int dx \frac{\delta}{\delta\psi_A^{\mu}(x,t)}F[\underrightarrow{\psi}] \times \frac{\partial\psi_A^{\mu}(x,t)}{\partial t}$$

- Apply fnal *integn by parts* gives eqn of motion for *phase space fnal average* of $F[\underrightarrow{\psi}(x,t)]$

$$\frac{\partial}{\partial t}\left\langle F[\underset{\rightarrow}{\psi}]\right\rangle_t =$$

$$\left\langle \sum_{\mu A} \int dx \{ \frac{\delta}{\delta\psi_A^{\mu}(x,t)} F[\underset{\rightarrow}{\psi}]\} \{A_A^{\mu}(x)\} \right\rangle$$

$$+\left\langle \begin{array}{c} \sum_{\mu A} \int dx \{ \frac{\delta}{\delta\psi_A^{\mu}(x,t)} F[\underset{\rightarrow}{\psi}]\} \\ \times \{\int dy \sum_{B\neq A} (\sum_{kl} \phi_{Ak}^{\mu}(x,t) C_{AkBl}^{\mu} \phi_{Bl}^{-\mu}(y,t))\, \psi_B^{\mu}(y,t)\} \end{array} \right\rangle$$

$$+\left\langle \begin{array}{c} \sum_{\mu A} \int dx \{ \frac{\delta}{\delta\psi_A^{\mu}(x,t)} F[\underset{\rightarrow}{\psi}]\} \\ \times \{\int dy \sum_{B\neq A} (\sum_{kl} \phi_{Bl}^{\mu}(x,t) C_{AkBl}^{-\mu} \phi_{Ak}^{-\mu}(y,t))\, \psi_A^{\mu}(y,t)\} \end{array} \right\rangle$$

$$+\left\langle \frac{1}{2} \sum_{\mu A, \nu B} \int\int dxdy \{ \frac{\delta}{\delta\psi_A^{\mu}(x,t)} \frac{\delta}{\delta\psi_B^{\nu}(y,t)} F[\underset{\rightarrow}{\psi}]\} \{E_{AB}^{\mu\nu}(x,y)\} \right.$$

- Extra terms involve *time dependent mode fns* and *coupling coefficients*.
- Extra terms involve *only* condensate to non-condensate mode couplings.

♦ Stochastic field eqns

$$\delta\psi_A^{\mu s}(\mathrm{x},\mathrm{t}) = G_A^{\mu}(\mathrm{x})\delta\mathrm{t} + \sum_a N_{Aa}^{\mu}(\mathrm{x}) \int_t^{t+\delta t} d\mathrm{t}_1 \Gamma_a(\mathrm{t}_1)$$

$$\frac{\partial}{\partial \mathrm{t}}\psi_A^{\mu s}(\mathrm{x},\mathrm{t}) = G_A^{\mu}(\mathrm{x}) + \sum_a N_{Aa}^{\mu}(\mathrm{x})\Gamma_a(\mathrm{t}_+)$$

- Variation $\delta\psi_A^{\mu s}(\mathrm{x},\mathrm{t}) = (\psi_A^{\mu s}(\mathrm{x},\mathrm{t}+\delta\mathrm{t}) - \psi_A^{\mu s}(\mathrm{x},\mathrm{t}))$
- Forms $G_A^{\mu}(\mathrm{x})$, $N_{Aa}^{\mu}(\mathrm{x})$ to be found.
- *Gaussian-Markoff* random noise $\Gamma_a(t)$

$$\overline{\Gamma_a(\mathrm{t}_1)} = 0$$

$$\overline{\Gamma_a(\mathrm{t}_1)\Gamma_b(\mathrm{t}_2)} = \delta_{ab}\delta(\mathrm{t}_1 - \mathrm{t}_2)$$

$$\overline{\Gamma_a(\mathrm{t}_1)\Gamma_b(\mathrm{t}_2)\Gamma_c(\mathrm{t}_3)} = 0$$

$$\begin{aligned}\overline{\Gamma_a(\mathrm{t}_1)\Gamma_b(\mathrm{t}_2)\Gamma_c(\mathrm{t}_3)\Gamma_d(\mathrm{t}_4)} &= \overline{\Gamma_a(\mathrm{t}_1)\Gamma_b(\mathrm{t}_2)}\,\overline{\Gamma_c(\mathrm{t}_3)\Gamma_d(\mathrm{t}_4)} \\ &+ \overline{\Gamma_a(\mathrm{t}_1)\Gamma_c(\mathrm{t}_3)}\,\overline{\Gamma_b(\mathrm{t}_2)\Gamma_d(\mathrm{t}_4)} \\ &+ \overline{\Gamma_a(\mathrm{t}_1)\Gamma_d(\mathrm{t}_4)}\,\overline{\Gamma_b(\mathrm{t}_2)\Gamma_c(\mathrm{t}_3)}\end{aligned}$$

- *Decorrelation* for function $H(\psi_A^{\mu s}(\mathrm{x},\mathrm{t}))$

$$\overline{H(\psi_A^{\mu s}(\mathrm{x},\mathrm{t}_1)\Gamma_a(\mathrm{t}_2)\Gamma_b(\mathrm{t}_3)\Gamma_c(\mathrm{t}_4)..\Gamma_k(\mathrm{t}_l)}$$

$$= \overline{H(\psi_A^{\mu s}(\mathrm{x},\mathrm{t}_1))}\;\overline{\Gamma_a(\mathrm{t}_2)\Gamma_b(\mathrm{t}_3)\Gamma_c(\mathrm{t}_4)..\Gamma_k(\mathrm{t}_l)} \qquad \mathrm{t}_1 < \mathrm{t}_2, \mathrm{t}_3, \ldots,$$

♦ Key step: stochastic average

- Change in *stochastic functional* $F[\underset{\rightarrow}{\psi}^{s}]$ due to *changes* $\delta\psi_{A}^{\mu s}(x,t)$ in *stochastic fields*

$$F[\underset{\rightarrow}{\psi}^{s}(x,t)+\delta\underset{\rightarrow}{\psi}^{s}(x,t)]-F[\underset{\rightarrow}{\psi}^{s}(x,t)]$$

$$=\int dx\sum_{\mu A}\delta\psi_{A}^{\mu s}(x,t)\left(\frac{\delta F[\underset{\rightarrow}{\psi}^{s}]}{\delta\psi_{A}^{\mu s}(x,t)}\right)_{x}$$

$$+\frac{1}{2}\int\int dxdy\sum_{\mu A,\nu B}\delta\psi_{A}^{\mu s}(x,t)\,\delta\psi_{B}^{\nu s}(y,t)\left(\frac{\delta^{2}F[\underset{\rightarrow}{\psi}^{s}]}{\delta\psi_{A}^{\mu s}(x,t)\delta\psi_{B}^{\nu s}(y,t)}\right)$$

- 1st term: stochastic average involves $G_{A}^{\mu}(x)$.
- 2nd term: stochastic average involves $N_{Aa}^{\mu}(x)$.
- Carry out stochastic averages using Gaussian-Markoff properties gives eqn of motion for *stochastic average* of $F[\underset{\rightarrow}{\psi}^{s}(x,t)]$

$$\frac{\partial}{\partial t}\overline{F[\underrightarrow{\psi}^{s}(x,t)]} =$$

$$\overline{\int dx \sum_{\mu A}\left(\frac{\delta F[\underrightarrow{\psi}^{s}]}{\delta\psi_{A}^{\mu s}(x,t)}\right)_{x} G_{A}^{\mu}(x)}$$

$$\overline{+\frac{1}{2}\int\int dxdy \sum_{\mu A,\nu B}\left(\frac{\delta^{2} F[\underrightarrow{\psi}^{s}]}{\delta\psi_{A}^{\mu s}(x,t)\,\delta\psi_{B}^{\nu s}(y,t)}\right)_{x,y}[[N(x)\ N^{T}(y)]_{A}^{\mu}}$$

♦ Relation Ito eqn and FFPE

- *KEY RESULT*
- For $\left\langle F[\underrightarrow{\psi}]\right\rangle_{t}$ and $\overline{F[\underrightarrow{\psi}^{s}(x,t)]}$ to be *same* for

arbitrary $F[\underrightarrow{\psi}]$ gives

$$G_{A}^{\mu}(x)=A_{A}^{\mu}(x)$$

$$+\int dy \sum_{B\neq A}\Big(\sum_{kl}\phi_{Ak}^{\mu}(x,t)C_{AkBl}^{\mu}\phi_{Bl}^{-\mu}(y,t)\Big)\ \psi_{B}^{\mu}(y,t)$$

$$+\int dy \sum_{B\neq A}\Big(\sum_{kl}\phi_{Bl}^{\mu}(x,t)C_{AkBl}^{-\mu}\phi_{Ak}^{-\mu}(y,t)\Big)\ \psi_{A}^{\mu}(y,t)$$

and

$$[N(x)\, N^T(y)]^{\mu,\nu}_{A,B}$$

$$= E^{\mu\nu}_{AB}(x,y)$$

$$= D^{\mu\nu}_{AB}(x,y)$$

$$+\frac{1}{2}\{\delta_{AC}\delta_{BNC}\,\delta_{\mu-\nu}\Big(\sum_k \sum_l \phi^{\mu}_{Ak}(x,t) C^{\mu}_{AkBl}(t)\phi^{\nu}_{Bl}(y,t)\Big)\}$$

$$+\frac{1}{2}\{\delta_{BC}\delta_{ANC}\,\delta_{\nu-\mu}\Big(\sum_k \sum_l \phi^{\nu}_{Bl}(y,t) C^{\nu}_{BlAk}(t)\phi^{\mu}_{Ak}(x,t)\Big)\}$$

- Existence of $N^{\mu}_{Aa}(x)$ depends on *factorisation* of diffusion matrix

$$[N(x)\, N^T(y)]^{\mu,\nu}_{A,B} = \sum_a N^{\mu}_{Aa}(x) N^{\nu}_{Ba}(y) = E^{\mu\nu}_{AB}(x,y)$$

- Determine $N^{\mu}_{Aa}(x)$ via K^{μ}_{Aka}

$$N^{\mu}_{Aa}(x) = \sum_k K^{\mu}_{Aka}(t)\, \phi^{\mu}_{Ak}(x,t)$$

$$E^{\mu\nu}_{AB}(x,y) = \sum_{kl} \phi^{\mu}_{Ak}(x,t)\, E^{\mu\nu}_{AkBl}(t)\, \phi^{\mu}_{Bl}(y,t)$$

Exists K^{μ}_{Aka} due *symmetry* $E^{\mu\nu}_{AkBl} = E^{\nu\mu}_{BlAk}$

$$\sum_a K^{\mu}_{Aka} K^{\nu}_{Bla} = E^{\mu\nu}_{AkBl}$$

Takagi factorisation (1925).

♦ Stochastic field interpretation

- Sum of *classical* and *noise* field terms

$$\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)=\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{class}+\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{noise}$$

$$\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{class}=G_A^{\mu}(x)$$

$$\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{noise}=\sum_a N_{Aa}^{\mu}(x)\,\Gamma_a(t_+)$$

- Stochastic averages: *classical field*

$$\overline{\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{class}}=0$$

- Non stochastic to all orders.
- Stochastic averages: *noise field*

$$\overline{\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x,t)\right)_{noise}}=0$$

$$\overline{\left(\frac{\partial}{\partial t}\psi_A^{\mu s}(x_1,t_1)\right)_{noise}\left(\frac{\partial}{\partial t}\psi_B^{\nu s}(x_2,t_2)\right)_{noise}}$$

$$=\delta(t_1-t_2)\;\overline{E_{AB}^{\mu\nu}(x_1,t_{1,2};x_2,t_{1,2})}$$

- All even orders non-zero stochastic averages related to *new diffusion matrix* $E_{AB}^{\mu\nu}$.

- Noise field *not* Gaussian-Markoff.
- Classical field and noise field terms are *both* related to standard *drift* $A_A^{\mu}(x)$ and *diffusion* terms $D_{AB}^{\mu\nu}(x, y)$ in the usual FFPE but in addition there are *extra terms* involving *time dependent mode fns* and *coupling coefficients*.
- Extra terms involve *only* condensate to non-condensate mode couplings.

CONCLUSION

- *Hybrid phase space theory* of single component BEC developed, where condensate modes treated via *Wigner* and non-condensate modes treated via *Positive P* distribution *functionals*.
- Theory treats case where *mode functions* are *time dependent*, as for *applications* in *BEC interferometry*.
- *Functional Fokker-Planck equation* has been obtained.
- *Drift terms* are *same* as in standard treatments with time *independent* modes.
- *Diffusion terms* contain *extra* contributions depending on time dependent *mode functions* and *coupling constants* involving integrals of mode functions and their *time derivatives*.
- Equivalent *Ito equations* for *stochastic condensate* and *non-condensate fields* are found, the fields are sum of *classica*l and *noise* fields.

- *Classical* fields given by *drift term* in FFPE, augmented by *extra* terms depending on time dependent *mode functions* and *coupling constants*.
- *Noise* fields related to *diffusion term* in FFPE in standard way, diffusion term containing *extra* contributions due to time dependent modes.
- Only coupling constants *between* condensate *and* non-condensate modes involved for both Ito and FFPE.
- *Stochastic properties* of *noise* fields given in terms of *diffusion matrix* in FFPE, with only stochastic averages of products of *even* numbers of noise fields are *non-zero.*
- Noise fields are *non Gaussian-Markoff*.