Two component Bose-Hubbard model with higher angular momentum states
We combine three areas of ultracold physics:
- ultracold dipolar gases
- spinor gases in a lattice (in the context of MI and SF transition)
- orbital superfluid

The main issue is to account for the spin degree of freedom as a dynamical variable in the lattice.

When spin dynamics takes place it could lead to the appearance of an orbital \((P_x + i P_y)\) superfluid.

Moreover it introduces an additional degree of control and leads to variety of different stable phases (PhD for small particle number).
Assumptions

- 2D square optical lattice with Cr atoms
- Limit basis to two states coupled by dipolar interaction at every lattice site

\[ m_S = 2 \; ; \; l = 1 \]
\[ \Psi_b(\vec{r}) \sim (w_1(x) w_0(y) + i \; w_0(x) w_1(y)) \; e^{-\frac{\Omega z^2}{2}} \]

\[ m_S = 3 \; ; \; l = 0 \]
\[ \Psi_a(\vec{r}) \sim w_0(x) w_0(y) \; e^{-\frac{\Omega z^2}{2}} \]

- Limiting subspace of essential states is a crucial approximation and it is possible only due to a weakness of dipolar interactions.
What’s the influence of weak dipolar interactions?

- Equilibration of the energy difference

\[ E_b - E_a \approx E_{dip} \]
Spin dynamic triggered by dipolar interactions

- There are several channels of dipolar collisions for two atoms. Fortunately we can choose the desired channel by a proper adjustment of the resonant external magnetic field.

\[
|2, 0\rangle \xrightarrow{D} |0, 2\rangle
\]

- \(m_s = 2; \quad l = 1\)
- \(m_s = 3; \quad l = 0\)
Hamiltonian of the system

\[
H = \sum_i (E_a + B) a_i^+ a_i + E_b b_i^+ b_i + \\
+ U_a a_i^+ a_i^+ a_i a_i + U_b b_i^+ b_i^+ b_i b_i + U_{ab} a_i^+ a_i^+ b_i b_i \\
+ D (a_i^+ a_i^+ b_i b_i + b_i^+ b_i^+ a_i a_i) + H_1(J_a, J_b)
\]

\[
H_1 = -J_a \sum_{\langle i, j \rangle} a_i^+ a_j - J_b \sum_{\langle i, j \rangle} b_i^+ b_j
\]
What’s the influence of weak dipolar interactions on 1 particle state per site?

- Equilibration of the energy difference
  \[ E_b - E_a \approx E_{dip} \]
- For 1 particle states average per site
  \[ |1,0\rangle \Leftrightarrow |0,1\rangle \]
- The lowest order process which contributes to the transfer between these state is a sequence of three events:
  \[
  |1,0\rangle \xrightarrow{J_a} |2,0\rangle \\
  |2,0\rangle \xrightarrow{D} |0,2\rangle \\
  |0,2\rangle \xrightarrow{J_b} |0,1\rangle 
  \]
Fisher method to find thermodynamically stable phases of the system in a chosen subspace

- In the Fisher method we assume for all sites that:
  \[ \langle a_i \rangle = \phi a_i \]
  \[ \langle b_i \rangle = \phi b_i \]

\[
H_1 = -J_a \sum_{\langle i,j \rangle} a_i^+ a_j - J_b \sum_{\langle i,j \rangle} b_i^+ b_j \rightarrow H_1 = -J_a \sum_i (a_i^+ \phi a_i + a_i \phi a_i^+) - J_b \sum_i (b_i^+ \phi b_i + b_i \phi b_i^+)
\]

- Boundaries between MI and SF are obtained from:

\[
\phi(a) = \lim_{\beta \to \infty} \frac{\text{Tr}[\hat{\mathcal{H}} e^{-\beta (H_0 + H_1)}]}{Z(\beta)}
\]

Z is the grand canonical partition function which reduces to a single lowest energy state contribution.

- To lowest order we get a linear and homogenous set of equations.
Single particle states in magnetic field

\[ \mu < 0 \quad \rightarrow \quad |0,0\rangle \]

\[ \mu > 0 \quad \rightarrow \quad |1,0\rangle \quad \text{or} \quad |0,1\rangle \]
Phase Diagram –

regions of stability of different possible phases of the system
Final conclusions

- Dipolar interactions can lead to novel phases, in particular to the appearance of orbital \((P_x + iP_y)\) superfluids in the \(b\) - component.

- The experiments with ultra weak magnetic fields with Cr atoms in the lattice are under extensive studies of B. Laburthe-Tolra group in Paris (PRA 81, 042716 (2010)).
THX
System is in superfluid phase (the mean occupation is fractional)

Large tunneling supports the ‘standard’ \( SF_a \) and orbital \( (P_x + i P_y) \) \( SF_b \).

When decreasing tunneling - particles enter \( SF_b \).

The grey area corresponds to the ‘stable vacuum’.
Region $0 < \frac{\mu}{U_a} < \frac{2 U_b}{U_a}$

- Large tunneling - $\text{SF}_{a+b}$
- Lower tunneling – $\text{MI}_a \ (n_a = 1)$, $\text{SF}_b$
- There is an additional stable phase – small region of the Mott insulator in the vortex component $\text{MI}_b \ (n_b = 0)$.

- Large tunneling - $\text{SF}_{a+b}$
- Lower tunneling – $\text{MI}_a \ (n_a = 0)$, $\text{SF}_b$
- There is an additional stable phase – small region of the Mott insulator in the vortex component $\text{MI}_b \ (n_b = 1)$. 
Final conclusions

- Even the case of one particle on average per site can introduce various novel phases to the system (especially the orbital superfluids in the excited energy state).
- Weak dipolar interaction can be resonantly tuned to couple the ground Wannier state to the excited one with orbital angular momentum.
- In future …
- When two particles occupy the same site, it is more favorable for dipolar transfer.
Phase Diagram -
regions of stability of different possible phases of the system
Region $\frac{\mu}{U_a} < 0$

- System is in superfluid phase (the mean occupation is fractional)
- Large tunneling supports the ‘standard’ $SF_a$ and orbital $(P_x + i P_y)$ $SF_b$.
- When decreasing tunneling - particles enter $SF_b$.
- The grey area corresponds to the ‘stable vacuum’.
Region \( 0 < \frac{\mu}{U_a} < \frac{2 U_b}{U_a} \)

- Large tunneling \(- SF_{a+b}\)
- Lower tunneling \(- SF_b\)
- There is an additional stable phase – small region of the Mott insulator in the vortex component \(MI_b\) (1 particle devide oneself like \(n_b = 1, n_a = 0\)).

Why doesn’t \(MI_a\) exist?
Why do particles choose to be in the b component?
Although the energies are equal at the resonance, larger tunneling in the vortex states favoure the b-component.
Final conclusions

- Even the case of one particle on average per site can introduce various novel phases to the system.
- Weak dipolar interaction can be resonantly tuned to couple the ground Wannier state to the excited one with orbital angular momentum.
- System realises the scenarios in which energy of the system is lower - favourable phases are the orbital \(( P_x + i P_y)\) superfluid or vortex Mott Insulator phases.
- In future ...
- When two particles occupy the same site, it is more favorable for dipolar transfer.