Phase Space Methods for Fermions using Grassmann Variables

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Abstract

In both quantum optics and cold atom physics, the behaviour of bosonic photons and atoms is often treated using phase space methods, where mode annihilation and creation operators are represented by c-number phase space variables, with the density operator equivalent to a distribution function of these variables. The anti-commutation rules for fermion annihilation, creation operators suggests the possibility of using anti-commuting Grassmann variables [1] to represent these operators. However, in spite of the seminal work by Cahill and Glauber [2] and a few applications [3, 4], the use of Grassmann phase space methods in quantum - atom optics to treat fermionic systems is rather rare, though fermion coherent states using Grassmann variables are widely used in particle physics.

The theory of Grassmann phase space methods for fermions is developed, showing how the distribution function is defined and used to determine quantum correlation functions via Grassmann phase space integrals, how the Fokker-Planck equations are obtained and then converted into equivalent Ito equations for stochastic Grassmann variables. The situation is somewhat different to the bosonic c-number case, the sign for the drift term in the Ito equation is reversed and the diffusion matrix in the Fokker-Planck equation is anti-symmetric rather than symmetric. Prospects for numerical calculations are discussed.

As a simple test case for future development of phase space Grassmann distribution methods for multi-mode fermionic applications, a positive P type distribution function involving both c-number variables (for the cavity mode) and Grassmann variables (for the two level atom) is used to treat the Jaynes-Cummings model [5]. The approach uses the canonical form of the (non-unique) positive P distribution function, and non-standard correspondence rules for bosonic operators lead to an unusual Fokker-Planck equation. A Fokker-Planck equation involving both left and right Grassmann differentiation is obtained for the distribution function, and which is equivalent to six coupled equations for the six c-number functions of the two bosonic variables that specify the Grassmann distribution function. Transformations to new bosonic variables rotating at the cavity frequency enables the six coupled equations for the new c-number functions to be solved analytically, based on an ansatz of Stenholm [6]. The distribution function is shown to be the same as that determined from the well-known solution based on coupled equations for state vector amplitudes of atomic and n-photon product states.

References

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